BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI

Tomul XXX (XXXIV)

Fasc. 1–4

Secţia I

MECANICĂ

1984

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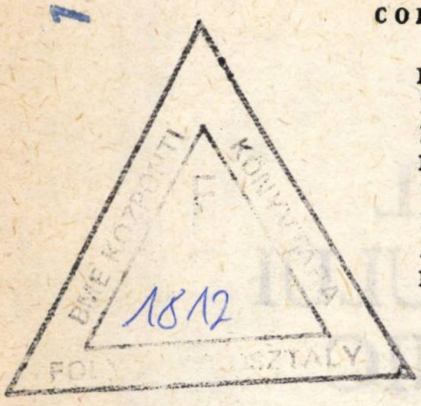

Această fascicolă a apărut şi cu sprijinul material al comitetului sindicatului şi consiliului U. A. S. C. al Institutului politehnic din Iaşi

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
BULLETIN OF THE POLYTECHNIC INSTITUTE OF JASSY
ИЗВЕСТИЯ ЯССКОГО ПОЛИТЕХНИЧЕСКОГО ИНСТИТУТА
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D.C. 007 ; 621.391

ON THE ENCODING IN CYBERNETIC SYSTEMS

BY

VALERIU B. MUNTEANU and M. T. LUCANU

1. The Definition of Information and Entropy in Cybernetic Systems. In cybernetic systems, the discrete complete sources without memory deliver messages with a certain probability, each message having a certain significance for the user [5].

If the importance of the received message is apreciated by means of a certain utility u_i, then the distribution of these sources can be written under the form

$$S: \begin{pmatrix} s_1 & s_2 & \dots & s_n \\ p_1 & p_2 & \dots & p_n \\ u_1 & u_2 & \dots & u_n \end{pmatrix}, \tag{1}$$

where S represents the complete discrete source without memory ; s_i are the messages of the source; u_i is the utility of s_i, appreciated by a natural number; p_i is the delivering probability of the message s_i, so that

$$p_i > 0, \quad i = \overline{1, n}; \tag{2}$$

$$\sum_{i=1}^{n} p_i = 1. \tag{3}$$

It is shown in [6], [7] that the information $I_{pu}(s_i)$ attached to a message s_i can be calculated with the relation

$$I_{pu}(s_i) = -\log_\alpha p_i + u_i, \tag{4}$$

where

$$\alpha \in (0,1) \vee (1, \infty). \tag{5}$$

The average information on the message, or the entropy of the source $H_{pu}(S)$ will be calculated then like the statistical average of the random variable $I_{pu}(s)$, i.e.

$$H_{pu}(S) = -\sum_{i=1}^{n} p_i \log_\alpha p_i + \sum_{i=1}^{n} p_i u_i. \tag{6}$$

2. Determination of the Average Length of Code Words. Let

(7) $$[X]=[x_1, x_2, \ldots, x_M],$$

be the alphabet of a code,

(8) $$[C]=[c_1, c_2, \ldots, c_n],$$

the words attached to the messages, and

(9) $$[L]=[l_1, l_2, \ldots, l_N],$$

the lengths of the code words.

Certainly, the probabilities and the utilities of the source messages S must equal the probabilities $p(c_i)$ and the utilities $u(c_i)$ of the code words, i.e.

(10) $$p_i=p(c_i),$$

(11) $$u_i=u(c_i).$$

If the probability with which the symbols x_k of the code alphabet are used in coding are $p(x_k)$, $k=\overline{1, M}$, then the average information on symbol, or the code entropy $H(X)$ can be calculated with the relation [1]

(12) $$H(X)=-\sum_{k=1}^{M} p(x_k) \log p(x_k).$$

The utilities of the code alphabet symbols are considered zero.

The utilities of the code words can be made evident by increasing their length.

The length l_i of a code word c_i can be then determined by the ratio between the information on the word and the average information on a symbol from the code alphabet, i.e.

(13) $$l_i=\frac{I_{pu}(s_i)}{H(X)},$$

or, taking into account (4), it results

(14) $$l_i=\frac{-\log_\alpha p_i+u_i}{H(X)}.$$

If

(15) $$\bar{l}=\sum_{i=1}^{n} p_i l_i,$$

where $\bar{l}$ is the average length of the code words, then taking into account (6), it follows

(16) $$\bar{l}=\frac{H_{pu}(S)}{H(X)}.$$

One can increase the transmission efficiency by conveniently assigning to each message s_i, a code word, so that the average length $\bar{l}$ should be small as possible.

Certainly, the transmission efficiency can only be defined if there is a lower boundary of $\bar{l}$.

One can see from (6) that $H_{pu}(S)$ can not be changed for a given source; in return, $H(X)$ will become maximum if [1], [2],

$$p(x_1)=p(x_2)=\ldots=p(x_M)=\frac{1}{M}. \tag{17}$$

In this case,

$$\max H(X)=\log_\alpha M. \tag{18}$$

If the symbols of the code alphabet are equiprobably used during the coding process, the average length of the code words becomes minimum $\bar{l}_{\min}$, i.e.

$$\bar{l}_{\min}=\frac{H_{pu}(S)}{\log_\alpha M}. \tag{19}$$

3. Optimal Coding in the General Case. The code efficiency η is defined by

$$\eta=\frac{\bar{l}_{\min}}{\bar{l}}, \tag{20}$$

or, taking into account (19), it follows

$$\eta=\frac{H_{pu}(S)}{\bar{l}\log_\alpha M}. \tag{21}$$

The codes having the unit efficiency are called absolutely optimal codes.

On the basis of (14), it results that, for an absolutely optimal code, the length of the code words must satisfy the condition

$$l_i=\frac{-\log p_i+u_i}{\log_\alpha M}, \quad i=\overline{1,n}. \tag{22}$$

On the other hand, the length of each code word must belong to the set N^*, since it consists of a integer of symbols from the code alphabet. That is why l_i must be chosen as the integer satisfying the condition

$$\frac{-\log_\alpha p_i+u_i}{\log_\alpha M}\leqslant l_i<\frac{-\log_\alpha p_i+u_i}{\log_\alpha M}+1. \tag{23}$$

If

$$\alpha=M, \tag{24}$$

with (24) and (23) it follows that the length of the code words l_i must satisfy the condition

$$p_i M^{-u_i}\geqslant M^{-l_i}, \quad i=\overline{1,n}, \tag{25}$$

or, equivalently

$$M^{-(l_i-u_i)}\leqslant p_i. \tag{26}$$

By summing up the relation (27) from 1 to n, it results

$$\sum_{i=1}^{n} M^{-(l_i-u_i)} \leqslant 1. \tag{27}$$

The relation (27) corresponds to the condition of existence of a unique code, decodable for the cybernetic systems.

If $u_i=0$, $i=\overline{1, n}$, the classical condition of existence is obtained of a unique decodable code, known in literature [1], [2], [4].

4. Extension of the Coding Theorem of Discrete Sources for Undisturbed Channels. By multiplying the relation (23) by p_i and summing up from 1 to n, it results

$$\frac{-\sum_{i=1}^{n} p_i \log_\alpha p_i + \sum_{i=1}^{n} p_i u_i}{\log_\alpha M} \leqslant \sum_{i=1}^{n} p_i l_i < \frac{-\sum_{i=1}^{n} p_i \log_\alpha p_i + \sum_{i=1}^{n} p_i u_i}{\log_\alpha M} + 1, \tag{28}$$

or, with (6) and (15)

$$\frac{H_{pu}(S)}{\log_\alpha M} \leqslant \bar{l} < \frac{H_{pu}(S)}{\log_\alpha M} + 1. \tag{29}$$

The relation (29) is valid for every discrete complete source without memory. It is also particularly valid for any extension of the source S [2]. If the coding is made on groups of m messages of S, one can get the coding of the mth extension of the given source. By noting by $H_{pu}(S^m)$ the entropy (of the mth extension of the source S, it results

$$H_{pu}(S^m) = mH_{pu}(S). \tag{30}$$

Indeed, let

$$\sigma_i = s_{i_1} \wedge s_{i_2} \wedge \ldots \wedge s_{i_m}, \quad i=\overline{1, n^m}, \tag{31}$$

be a composite symbol of the mth extension of the source S, where $s_{i_1}, s_{i_2}, \ldots, s_i$ are the messages of the source.

The source S being without memory, it results

$$p(\sigma_i) = p(s_{i_1})p(s_{i_2}) \ldots p(s_{i_m}), \tag{32}$$

$$u(\sigma_i) = u(s_{i1}) + u(s_{i_2}) + \ldots + u(s_{i_m}). \tag{33}$$

The entropy of the mth extension of the source S will be calculated, according to (6), from the relation

$$H_{pu}(S^m) = -\sum_{i=1}^{n^m} p(\sigma_i) \log_\alpha p(\sigma_i) + \sum_{i=1}^{n^m} p(\sigma_i) u(\sigma_i). \tag{34}$$

Let have a symbol made of $m+1$ messages of the source S, under the form

$$\xi_k = \sigma_i \wedge s_j, \quad i=\overline{1, n^m}, \quad j=\overline{1, n}, \tag{35}$$

where s_j is a message of the source S.

Since

$$p(\xi_k)=p(\sigma_i)p(s_j), \tag{36}$$

$$u(\xi_k)=u(\sigma_i)+u(s_j), \tag{37}$$

$$\sum_{i=1}^{n^m} p(\sigma_i)=1, \tag{38}$$

$$\sum_{j=1}^{n} p(s_j)=1, \tag{39}$$

the entropy of the $(m+1)^{\text{th}}$ extension becomes

$$H_{pu}(S^{m+1})=-\sum_{k=1}^{n^{m+1}} p(\xi_k)\log_\alpha p(\xi_k)+\sum_{k=1}^{n^{m+1}} p(\xi_k)u(\xi_k), \tag{40}$$

or, taking into account (36)...(39), it follows

$$\begin{aligned} H_{pu}(S^{m+1})=&-\sum_{i=1}^{n^m} p(\sigma_i)\log_\alpha p(\sigma_i)+\sum_{i=1}^{n^m} p(\sigma_i)u(\sigma_i)-\\ &-\sum_{j=1}^{n} p(s_j)\log_\alpha p(s_j)+\sum_{j=1}^{n} p(s_j)u(s_j), \end{aligned} \tag{41}$$

which is equivalent to the relation (30).

The mth extension of the source S being it self a discrete complete source without memory, it results, according to (29),

$$\frac{H_{pu}(S^m)}{\log_\alpha M}\leqslant \bar{l}_m<\frac{H_{pu}(S^m)}{\log_\alpha M}+1, \tag{42}$$

where $\bar{l}_m$ is the average length of a code word, corresponding to a series of m messages of the source S.

Since

$$\bar{l}=\frac{\bar{l}_m}{m}, \tag{43}$$

it results

$$\frac{H_p(S)}{\log_\alpha M}\leqslant \bar{l}<\frac{H_{pu}(S)}{\log_\alpha M}+\frac{1}{m}. \tag{44}$$

From the relation (44) it follows that, at the limit, when $m\to\infty$, the average length of the code words becomes equal to the minimum average length. Thus, the relation (44) becomes a generalization of C. E. Shannon's theorem [3], for encoding the discrete sources for undisturbed channels, in the case of the cybernetic systems. If the utilities of the messages of source S are zero, the classical results will be obtained [1]...[3].

5. Conclusions. By extending the notion of entropy to cybernetic systems, the average length of the code words was calculated, by generaliz-

ing the existence theorem of the uniquely decodable codes and the C.E. Shannon's theorem concerning the coding of discrete sources for undisturbed channels.

Received February 29, 1984

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REFERENCES

1. Gallager R.G., *Information Theory and Reliable Communication*. John Wiley and Sons, Inc., New York, 1968.
2. Spătaru Al., *Teoria informaţiei*. Ed. tehn., Buc., 1971.
3. Shannon C.E., *A Mathematical Theory of Communication*. Bell Systems Tech. Journal, 1948.
4. Mc Millan B., *The Basic Theorems of Information Theory*. Ann. Math. Stat., *24* (1953).
5. Beliş M., *A Quantitative-qualitative Measure of Information in Cybernetic Systems*. IEEE Trans. Inf. Theory, **14** (1968).
6. Munteanu V. u.a., *Über die Messung qualitativer Nachricht*. Bul. Inst. polit. Iaşi, **XXVII** (**XXXI**), *3–4*, s. III, 83–91 (1981).
7. Munteanu V. a.o., *The Entropy of the Cybernetic Systems*. Electrotehnica – Electronica – Automatica (Bucureşti), *2* (1981).

ASUPRA CODĂRII ÎN SISTEMELE CIBERNETICE

(Rezumat)

Extinderea noţiunii de entropie pentru sistemele cibernetice a permis calculul lungimii medii a cuvintelor de cod generalizîndu-se astfel teorema de existenţă a codurilor unic decodabile. De asemenea a fost generalizată prima teoremă a lui C.E. Shannon pentru sistemele cibernetice.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4 1984
Secţia I
MECANICĂ

D.C. 517.43 ; 512.9

INTEGRAL TRANSFORMS OF FOURIER-TYPE ON COMMUTATIVE ALGEBRAS

BY

DANIEL CONDURACHE

1. Let be $\mathcal{A}$ a direct product of $n+1$ algebras over the real number field $\mathcal{A}_p$, $p=\overline{0, n}$, isomorphic with the complex field.

Let $\hat{1}_p$ and $\hat{\imath}_p$ be elements of a basis in $\mathcal{A}_p$ with the multiplication table

$$\hat{1}_p\hat{\imath}_p=\hat{\imath}_p\hat{1}_p, \qquad \hat{\imath}_p^2=-\hat{1}_p, \qquad \hat{1}_p^2=\hat{1}_p. \tag{1}$$

The elements in $\mathcal{A}$ denoted as

$$v_p=\hat{\imath}_0^{p_0}\hat{\imath}_1^{p_1}\dots\hat{\imath}_n^{p_n}, \qquad p=\overline{0, 2^{n+1}-1} \tag{2}$$

(where $p=\overline{p_0 p_1 \dots p_{n(2)}}$ represent the writing of p in a binary system with n bits, supposing :

$$\hat{\imath}_k^{p_k}=\begin{cases}\hat{1}_k & \text{if} \quad p_k=0,\\ \hat{\imath}_k & \text{if} \quad p_k=1\end{cases}$$

represent a basis in $\mathcal{A}$.

With (2), the multiplication table in $\mathcal{A}$ is well defined by (1). Algebra $\mathcal{A}$ is of an $n+1$ order and it is also associative and commutative with unit

$$\hat{v}_0=\hat{1}_0\hat{1}_1\dots\hat{1}_n \overset{\text{not}}{=} \hat{1}. \tag{3}$$

$\mathbb{R}$ corpus shall be identified with the subalgebra generated by $\{\xi_i\hat{1}_0\}_{\xi_i\in\mathbb{R}}$ while $\mathbb{C}$ complex corpus shall be identified with subalgebra $\{\lambda 1_0+\mu\hat{\imath}_0\}_{\lambda,\mu\in\mathbb{R}}$.

We shall denote as $\mathbb{L}_1=\left\{f:\mathbb{R}\to\mathbb{R},\ \text{measurable and} \int_{-\infty}^{\infty}|f(t)|dt<\infty\right\}$

and $\mathbb{F}=\{\hat{f}:\mathbb{R}^{n+1}\to\mathcal{A}\}$ the set of functions of $n+1$ real variables taking values in $\mathcal{A}$.

For $\hat{w}\in\mathcal{A}$, $e^{\hat{w}}$ is defined as

$$e^{\hat{w}}=\sum_{k=0}^{\infty}\frac{\hat{w}^k}{k!}. \tag{4}$$

Transformation $\widehat{\mathbb{F}}_f : \mathbb{L}_1 \to \mathbb{F}$ is defined, for any function $f \in \mathbb{L}_1$ as it follows :

$$\widehat{\mathbb{F}}_f(\hat{\omega}) = \int_{-\infty}^{\infty} f(t) e^{-\hat{\omega} t} dt \, ; \quad \hat{\omega} = \sum_{j=0}^{n} \omega_j \hat{\imath}_j, \quad \omega_j \in \mathbb{R}, \quad j = \overline{0, n}. \tag{5}$$

Denotations

$$\Omega_k = \omega_0 + \sum_{\nu=1}^{n} (-1)^{k_\nu} \omega_\nu \, , \qquad k = \overline{0, 2^n - 1}, \tag{6}$$

$$e_k = \frac{1}{2^n} \prod_{\nu=1}^{n} [\hat{1} - (-1)^{k_\nu} \hat{\imath}_0 \hat{\imath}_\nu], \tag{7}$$

where $k = \overline{k_1 k_2 \ldots k_{n(2)}}$ is the writing in a bynary system with n bits of number k, shall be used.

Fourier transform of function $f \in \mathbb{L}_1$, defined over algebra $\mathbb{C}$ shall be denoted as

$$\mathcal{F}_f(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i_0 \omega t} dt, \qquad \omega \in \mathbb{R}. \tag{8}$$

2. Theorem 1. *Transform* (5) *exists for any function in* $\mathbb{L}_1$ *and is monogenic if considered as function of a hypercomplex variable on the commutative algebra* $\mathcal{A}$. *The unique decomposition*

$$\widehat{\mathbb{F}}_f = \sum_{k=0}^{2^n - 1} \mathcal{F}_f(\Omega) \hat{e}_k \tag{9}$$

takes place.

Proof. Due to (1), Euler-type relations are established in $\mathcal{A}$:

$$e^{-i_p \omega_p t} = \cos \omega_p t - \hat{\imath}_p \sin \omega_p t.$$

As it follows,

$$e^{-\hat{\omega} t} = e^{-\sum_{p=0}^{n} \hat{\imath}_p \omega_p t} = \prod_{p=0}^{n} e^{-\hat{\imath}_p \omega_p t} = \prod_{p=0}^{n} (\cos \omega_p t - \hat{\imath}_p \sin \omega_p t) =$$

$$= \sum_{p=0}^{2^{n+1}-1} \varphi_p(t, \omega_0, \omega_1, \ldots, \omega_n) \hat{v}_p,$$

where $\hat{v}_p$, $p = \overline{0, 2^{n+1} - 1}$ are (2) elements of basis and

$$\varphi_p = \prod_{\nu=0}^{n} \cos\left(\omega_0 t - p_0 \frac{\pi}{2}\right), \qquad p = \overline{p_0 p_1 \ldots p_{n(2)}}. \tag{10}$$

Accordingly (10)

$$|\varphi_p| < 1 \qquad \forall (t, \omega_0, \omega_1, \ldots, \omega_n) \subset \mathbb{R}^{n+2}.$$

Relation (5) becomes

$$\widehat{\mathbb{F}}_f=\int_{-\infty}^{\infty}\sum_{k=0}^{2^{n+1}-1} f(t)\varphi_k(t)\,\hat{v}_k=\sum_{k=0}^{2^{n+1}-1}\left(\int_{-\infty}^{\infty} f(t)\varphi_k(t)\mathrm{d}t\right)\hat{v}_k.$$

For any $f\in\mathbb{L}_1$, $\widehat{\mathbb{F}}_f$ components exist as with (10) we have

$$\left|\int_{-\infty}^{\infty} f(t)\varphi_k(t)\mathrm{d}t\right|<\int_{-\infty}^{\infty}|f(t)|\,\mathrm{d}t<\infty.$$

The monogenity is proved by calculating the exterior product which vanish identically in $\mathcal{A}$, due to the monogenitiy of the exponential on algebra $\mathcal{A}$.

Relation (9) is to be proved by means of mathematical induction. For $n=1$ a result demonstrated in [1] is obtained.

Let $\widehat{\mathbb{F}}_f^{(n+1)}$ be transform (4) defined on algebra $\mathcal{A}_{n+1}$, direct product between algebra $\mathcal{A}$ and the algebra isomorphic with the complex field generated by $\{\lambda\hat{\imath}_{n+1}+\mu\hat{\imath}_{n+1}\}$, $\hat{\imath}_{n+1}^2=-\hat{1}_{n+1}$, i.e. algebras—direct product of $n+2$ algebras isomorphic with the complex field.

Succesively, one obtains

$$\widehat{\mathbb{F}}_f^{(n+1)}=\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-(\hat{\imath}_0\omega_0+\ldots+\hat{\imath}_{n+1}\omega_{n+1})t}\,\mathrm{d}t=\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\hat{\omega}t-\hat{\imath}_{n+1}\omega_{n+1}t}\,\mathrm{d}t= \tag{11}$$

$$=\int_{-\infty}^{\infty} f(t)\cos\omega_{n+1}t\mathrm{e}^{-\hat{\omega}t}\mathrm{d}t-\hat{\imath}_{n+1}\int_{-\infty}^{\infty} f(t)\sin\omega_{n+1}t\mathrm{e}^{-\hat{\omega}t}\,\mathrm{d}t=\widehat{\mathbb{F}}_{f\cos\omega_{n+1}t}-\hat{\imath}_{n+1}\widehat{\mathbb{F}}_{f\sin\omega_{n+1}t}.$$

Supposing (9) as true, relation (11) becomes

$$\widehat{\mathbb{F}}_f^{(n+1)}=\sum_{k=0}^{2^{n+1}-1}\mathcal{F}_{f\cos\omega_{n+1}t}(\Omega_k)\hat{\mathrm{e}}_k-\hat{\imath}_{n+1}\sum_{k=0}^{2^{n+1}-1}\mathcal{F}_{f\sin\omega_{n+1}t}(\Omega_k)_k\hat{\mathrm{e}}. \tag{12}$$

Using the already known characteristics of Fourier transform, relation (12) may be written

$$\widehat{\mathbb{F}}_f^{(n+1)}=\sum_{k=0}^{2^{n+1}-1}\frac{\mathcal{F}_f(\Omega_k-\omega_{n+1})+\mathcal{F}_f(\Omega_k+\omega_{n+1})}{2}\hat{\mathrm{e}}_k-$$

$$-\hat{\imath}_{n+1}\sum_{k=0}^{2n+1-1}\frac{\mathcal{F}_f(\Omega_k-\omega_{n+1})-\mathcal{F}_f(\Omega_k+\omega_{n+1})}{2\hat{\imath}_0}\hat{\mathrm{e}}_k= \tag{13}$$

$$=\sum_{k=0}^{2^{n+1}-1}\mathcal{F}_f(\Omega_k-\omega_{n+1})\frac{1+\hat{\imath}_{n+1}\hat{\imath}_0}{2}\hat{\mathrm{e}}_k+\sum_{k=0}^{2^{n+1}-1}\mathcal{F}_f(\Omega_k+\omega_{n+1})\frac{1-\hat{\imath}_0\hat{\imath}_{n+1}}{2}\hat{\mathrm{e}}_k.$$

It is easy to notice that, among denotations (6) and (7) and the similar ones for the direct product of $n+2$ algebras, the following relations are established :

$$\Omega_{2k}^{(n+1)}=\Omega_k+\omega_{n+1}, \qquad \hat{e}_{2k}^{(n+1}=\hat{e}_k\,\frac{\hat{1}_{n+1}-\hat{1}_0\hat{1}_{n+1}}{2},$$

$$\Omega_{2k+1}^{(n+1)}=\Omega_k-\omega_{n+1}, \qquad \hat{e}_{2k+1}^{(n+1)}=\hat{e}_k\,\frac{\hat{1}_{n+1}+\hat{1}_0\hat{1}_{n+1}}{2}.$$

Consequently, relation (13) becomes

$$\widehat{\mathbb{F}}_j^{(n+1)}=\sum_{k=0}^{2^{n+1}-1}\mathcal{F}_f(\Omega_{2k+1}^{(n+1)})\hat{e}_{2k+1}^{(n+1)}+\sum_{k=0}^{2^{n+1}-1}\mathcal{F}_f(\Omega_{2k}^{(n+1)})\hat{e}_{2k}^{(n+1)}=\sum_{r=1}^{2^{n+2}-1}\mathcal{F}_f(\omega_r^{(n+1)})\hat{e}_r^{(n+1)},$$

which proves the validity of (9) for any $n\in\mathbb{N}$.

The uniqueness of the decomposition results from the following

L e m m a. *Elements* $\hat{e}_k$, $k=\overline{0, 2^n-1}$, *represent* 2^n *orthogonal idempotents in algebra* $\mathcal{A}$.

P r o o f. We have

$$\hat{e}_k\hat{e}_p=\frac{1}{2^{2n}}\prod_{\nu=1}^{n}[\hat{1}-(-1)^{k_\nu}\hat{1}_0\hat{1}_\nu][\hat{1}-(-1)^{p_\nu}\hat{1}_0\hat{1}_\nu]=$$

$$(14)\qquad =\frac{1}{2^{2n}}\prod_{\nu=0}^{n}\{\hat{1}+(-1)^{k_\nu+p_\nu}-\hat{1}_0\hat{1}_\nu][(-1)^{k_\nu}+(-1^{p_\nu})]\}.$$

From (14), if $k=p$ ($k_\nu=p_\nu$, $\forall\nu=\overline{1, n}$), it results $\hat{e}_k^2=\hat{e}_k$ and if $k\neq p$ ($\exists\nu$ so that $k_\nu\neq p_\nu$), it results $\hat{e}_k\hat{e}_p=\hat{0}$.

Notice. 1. It is easy to establish that the set $\{\hat{e}_k, \hat{1}_0\hat{e}_k\}_{k=\overline{0, 2^\lambda-1}}$, is constituted of linear independent elements, so that may form a basis in. algebra $\mathcal{A}$. A certain element from $\mathcal{A}$ is written in this basis as

$$(15)\qquad \hat{w}=\sum_{k=0}^{2^n-1}(a_k\hat{e}_k+b_k\hat{1}_0\hat{e}_k)=\sum_{k=0}^{2^n-1}(a_k+b_k\hat{1}_0)\hat{e}_k)\ ;\quad a_k,\ b_k\in\mathbb{R},\quad k=\overline{0, 2^n-1}$$

The unit of algebra in this basis is

$$\hat{1}=\sum_{k=0}^{2^n-1}\hat{e}_k.$$

Taking into account the demonstrated Lemma and the already known properties of algebras over the real number field, relation (15) proves algebra $\mathcal{A}$ to be a polynominal, quasi-simple one, being the direct sum of 2^n algebras isomorphic with the complex field.

3. Transform (5) has a remarkable property given by

T h e o r e m 2. *Integral transform* (5) *is an invertible one, its inverse being*

$$(16)\qquad \frac{f(t+0)+f(t-0)}{2}=\widehat{\mathbb{F}}^{-1}(\hat{\omega})=\frac{1}{2\pi\hat{1}_0}\int_{\mathcal{A}}\widehat{\mathbb{F}}_f(\hat{\omega})e^{\hat{\omega}t}\,d\hat{\omega}.$$

P r o o f. Relation

$$\hat{\omega}=\sum_{k=0}^{2^n-1}\Omega_k\hat{\imath}_0\hat{e}_k \tag{17}$$

is easy to be proved by means of mathematical induction, following a reasoning similar with the one from § 2.

Though $\hat{e}_k$, $k=\overline{0,2^n-1}$, are orthogonal indempotents, from (4) and (7) it results

$$e^{\hat{\omega}t}=\sum_{k=0}^{2^n-1}e^{\hat{\imath}_0\Omega_k t}\hat{e}_k. \tag{18}$$

Applying (9), (17) and (18) and the inverting formula of Fourier transform, we succesively obtain

$$\frac{1}{2\pi\hat{\imath}_0}\int_{\mathcal{A}}\widehat{\mathbb{F}}_f(\hat{\omega})e^{\hat{\omega}t}d\hat{\omega}=\frac{1}{2\pi\hat{\imath}_0}\int_{\mathcal{A}}\left(\sum_{k=0}^{2^n-1}\mathcal{F}_f(\Omega_k)\hat{e}_k\right)\left(\sum_{k=0}^{2^n-1}e^{\hat{\imath}_0\Omega_k t}\right)d\left(\sum_{k=0}^{2^n-1}\Omega_k\hat{\imath}_0\hat{e}_k\right)=$$

$$=\sum_{k=0}^{2^n-1}\frac{\hat{e}_k}{2\pi}\left(\int_{-\infty}^{+\infty}\mathcal{F}_f(\Omega_k)e^{\hat{\imath}_0\Omega_k t}d\Omega_k\right)=\sum_{k=0}^{2^n-1}\frac{f(t+0)+f(t-0)}{2}\hat{e}_k=$$

$$=\frac{f(t+0)+f(t-0)}{2}\sum_{k=0}^{2^n-1}\hat{e}_k=\frac{f(t+0)+f(t-0)}{2}.$$

Applications of the integral transform suggested in the present paper concerning the linear and non-linear dynamic systems will be subject to further work.

Received September 16, 1984

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REFERENCES

1. C o n d u r a c h e D., *Integral Transforms on Algebras* (I). Bul. Inst. polit. Iaşi, **XXVIII (XXXII)**, *1–4*, s. III (1982).
2. C o n d u r a c h e D., *Integral Transforms on Algebras* (II). Bul. Inst. polit. Iaşi, **XXX(XXXIV)**, *1–4*, s. III (1984).
3. R o ş c u l e ţ M., *Funcţii monogene pe algebre comutative*. Ed. Acad. R.S.R., Buc., 1985.

TRANSFORMĂRI INTEGRALE DE TIP FOURIER PE ALGEBRE COMUTATIVE

(Rezumat)

Se propune o transformare integrală de tip Fourier pe o algebră comutativă, produs direct a $n+1$ algebre complexe. Se demonstrează în condiţii foarte generale inversabilitatea acestei transformări.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

C.D. 531.1

SUR L'INVARIABILITÉ DE L'AXE INSTANTANÉ DE ROTATION

I. LE CAS DES DEUX REPÈRES CARTÉSIENS TRIORTHOGONAUX SOLIDAIRES D'UN MÊME CORPS RIGIDE

PAR

C. ACKER

Il a y déjà quelques années depuis qu'on a mis la question si la position de l'axe instantané de rotation dépend ou non de la position du repère mobile cartésien triorthogonal, solidarisé au corps rigide, qu'on utilise. Ci-dessous, nous présentons quelques argumentations à cet égard.

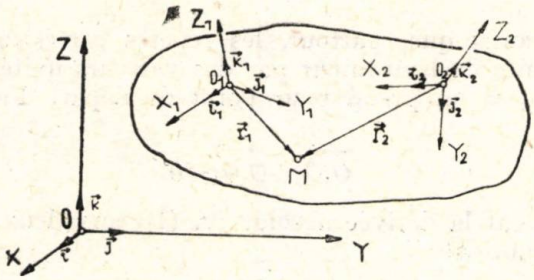

Fig. 1

Nous allons considérer connues :

a) l'expression de la dérivée absolue d'un vecteur qui dépend du temps t et dont l'expression analytique est donnée (v. la fig. 1) par rapport à un repère mobil $X_1O_1Y_1Z_1$, ou $X_2O_2Y_2Z_2$, cartésien triorthogonal, solidarisé au corps rigide, par exemple

$$\frac{d\mathbf{r}_1}{dt} = \frac{\partial^* \mathbf{r}_1}{\partial t} + \mathbf{w}_1 \times \mathbf{r}_1, \tag{1}$$

où $\partial^* \mathbf{r}_1/\partial t$ représente la dérivée par rapport au temps t du vecteur $\mathbf{r}_1$ relative au repère $X_1O_1Y_1Z_1$, $\mathbf{w}_1$ est le vecteur vitesse angulaire du repère $X_1O_1Y_1Z_1$ et $d\mathbf{r}_1/dt$ est la dérivée absolue par rapport au temps t de $\mathbf{r}_1$, aussi bien que

b) l'équation vectorielle de l'axe instantané de rotation d'un corps rigide, par rapport au repère mobile $X_1O_1Y_1Z_1$ (v. la fig. 1) solidarisé au corps rigide,

$$\mathbf{r}_1 = \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{\mathbf{w}_1^2} + \lambda \mathbf{w}_1 \tag{2}$$

($\mathbf{v}_{01}$ est la vitesse de l'origine O_1), ou par rapport au repère mobile $X_2O_2Y_2Z_2$, qui est solidaire du même corps rigide,

$$\mathbf{r}_2 = \frac{\mathbf{w}_2 \times \mathbf{v}_{O_2}}{\mathbf{w}_2^2} + \lambda \mathbf{w}_2 \tag{3}$$

($\mathbf{w}_2$ est le vecteur vitesse angulaire du repère $X_2O_2Y_2Z_2$ et $\mathbf{v}_{O_2}$ est la vitesse de l'origine O_2), ou par rapport au repère fixe $XOYZ$ (v. la fig. 1), soit qu'on utilise pour cela le repère mobile $X_1O_1Y_1Z_1$, solidaire du corps rigide, c'est à dire

$$\mathbf{r} = \mathbf{r}_{O_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{\mathbf{w}_1^2} + \lambda \mathbf{w}_1 \tag{4}$$

($\mathbf{r}_{O_1}$ est le vecteur de la position de l'origine O_1, par rapport au repère fixe $XOYZ$), soit qu'on utilise pour cela le repère mobile $X_2O_2Y_2Z_2$, solidaire du même corps rigide, c'est à dire

$$\mathbf{r} = \mathbf{r}_{O_2} + \frac{\mathbf{w}_2 \times \mathbf{v}_{O_2}}{\mathbf{w}_2^2} + \lambda \mathbf{w}_2 \tag{5}$$

($\mathbf{r}_{O_2}$ est le vecteur de la position de l'origine O_2, par rapport au repère fixe $XOYZ$).

Signalons aussi que, partout, les repères cartésiens triorthegonaux utilisés sont formés exclusivement par des vecteurs unités.

Maintenant, si on prend pour point de départ l'identité évidente (v. la fig. 1)

$$\overrightarrow{O_1O_2} + \overrightarrow{O_2O_1} = 0 \tag{6}$$

alors, en appliquant la dérivée absolue (v. (1)) aux deux membres de la relation (6), on obtient

$$\frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{O_1O_2} + \frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{O_2O_1} = 0, \tag{7}$$

où le premier terme du premier membre on considère être la dérivée absolue effectuée par l'utilisation du repère $X_1O_1Y_1Z_1$ (v. la fig. 1) en considérant aussi pour le vecteur $\overrightarrow{O_1O_2}$ l'expression analytique rapportée au même repère $X_1O_1Y_1Z_1$ —, le deuxième terme du premier membre représentant la dérivée absolue effectuée par l'utilisation du repère $X_2O_2Y_2Z_2$ (v. la fig. 1)—en considérant pour le vecteur $\overrightarrow{O_2O_1}$ l'expression analytique relative au repère $X_2O_2Y_2Z_2$.

En observant que les projections orthogonaux du vecteur $\overrightarrow{O_1O_2}$ sur les axes du repère $X_1O_1Y_1Z_1$, aussi que les projections orthogonaux du vecteur $\overrightarrow{O_2O_1}$ sur les axes du repère $X_2O_2Y_2Z_2$, ne dépendent pas du temps (et cela parce que O_1 et O_2 sont des points—ou des particules—appartenant au corps rigide, en même temps que les repères $X_1O_1Y_1Z_1$ et $X_2O_2Y_2Z_2$ sont solidaires du corps rigide, donc solidaires l'un de l'autre), il suit

(8) $$\frac{d}{dt}\overrightarrow{O_1O_2}=\mathbf{w}_1\times\overrightarrow{O_1O_2},$$

respectivement

(9) $$\frac{d}{dt}\overrightarrow{O_2O_1}=\mathbf{w}_2\times\overrightarrow{O_2O_1}.$$

Une conséquence de l'identité (6) est donc

(10) $$\mathbf{w}_1\times\overrightarrow{O_1O_2}+\mathbf{w}_2\times\overrightarrow{O_2O_1}=0,$$

ou

(11) $$(\mathbf{w}_1-\mathbf{w}_2)\times\overrightarrow{O_1O_2}=0,$$

cette relation (11) pouvant être obtenue par des différents autres procédés, assez nombreux, par exemple en exprimant la vitesse du point O_1 par

(12) $$\mathbf{v}_{O_1}=\mathbf{v}_{O_2}+\mathbf{w}_2\times\overrightarrow{O_2O_1},$$

et la vitesse du point O_2 par

(13) $$\mathbf{v}_{O_2}=\mathbf{v}_{O_1}+\mathbf{w}_1\times\overrightarrow{O_1O_2},$$

ce qui permet de présenter la différence $\mathbf{v}_{O_2}-\mathbf{v}_{O_1}$ par les expressions

(14) $$\mathbf{v}_{O_2}-\mathbf{v}_{O_1}=\mathbf{w}_2\times\overrightarrow{O_1O_2}$$

et

(15) $$\mathbf{v}_{O_2}-\mathbf{v}_{O_1}=\mathbf{w}_1\times\overrightarrow{O_1O_2},$$

ces deux relations conduisant facilement à la relation (11).

On doit souligner le fait que la relation (11) est satisfaite d'une manière identique, à cause de la relation

(16) $$\mathbf{w}_1=\mathbf{w}_2,$$

qui est identiquement satisfaite lorsque les repères $X_1O_1Y_1Z_1$ et $X_2O_2Y_2Z_2$ sont solidaires l'un de l'autre, en particulier lorsque les deux repères sont solidaires d'un même corps rigide, tel que nous allons le démontrer plus bas, ne devant pas considérer les relations $\mathbf{w}_1=\mathbf{w}_2$, ou $\overrightarrow{O_1O_2}=0$, ou $\mathbf{w}_1-\mathbf{w}_2=$ $=\mu\overrightarrow{O_1O_2}$ comme impliquées par la relation (11), particulièrement la dernière de ces trois relations ne devant pas être considérée comme conséquence de la relation (11), cette 3-ème relation de plus haut semblant être un des motifs de l'apparition du „problème" indiqué au début de la présente Note.

Démontrons donc le suivant

Théorème. *Si les repères cartésiens triorthogonaux $X_1O_1Y_1Z_1$ et $X_2O_2Y_2Z_2$ sont solidaires l'un de l'autre (en particulier lorsque les deux repères sont solidaires d'un même corps rigide), alors les respectifs vecteurs vitesses angulaires sont égaux.*

D é m o n s t r a t i o n. a) Lorsque le repère $X_2O_2Y_2Z_2$ s'obtient de $X_1O_1Y_1Z_1$ par une translation, alors les relations

$$\mathbf{i}_2=\mathbf{i}_1,\quad \mathbf{j}_2=\mathbf{j}_1,\quad \mathbf{k}_2=\mathbf{k}_1 \tag{17}$$

sont satisfaites, ce qui entraîne (en appliquant la dérivée absolue aux deux membres de cha une des trois relations précédentes)

$$\dot{\mathbf{i}}_2=\dot{\mathbf{i}}_1,\quad \dot{\mathbf{j}}_2=\dot{\mathbf{j}}_1,\quad \dot{\mathbf{k}}_2=\dot{\mathbf{k}}_1, \tag{18}$$

les relations (17) et (18) impliquant

$$\left\{\begin{aligned} w_{x_2}&=\dot{\mathbf{j}}_2.\mathbf{k}_2=\dot{\mathbf{j}}_1.\mathbf{k}_1=w_{x_1},\\ w_{y_2}&=\dot{\mathbf{k}}_2.\mathbf{i}_2=\dot{\mathbf{k}}_1.\mathbf{i}_1=w_{y_1},\\ w_{z_2}&=\dot{\mathbf{i}}_2.\mathbf{j}_2=\dot{\mathbf{i}}_1.\mathbf{j}_1=w_{z_1}\,; \end{aligned}\right. \tag{19}$$

alors, de (17) et de $\mathbf{w}_1=w_{x_1}\mathbf{i}_1+w_{y_1}\mathbf{j}_1+w_{z_1}\mathbf{k}_1$ et $\mathbf{w}_2=w_{x_2}\mathbf{i}_2+w_{y_2}\mathbf{j}_2+w_{z_2}\mathbf{k}_2$, il suit

$$\mathbf{w}_1=\mathbf{w}_2. \tag{20}$$

b) Si le repère $X_2O_2Y_2Z_2$ s'obtient de $X_1O_1Y_1Z_1$ par une rotation, ou par une translation suivie par une rotation, ou vice versa, par une rotation suivie par une translation, alors les expressions analytiques des vecteurs unités $\mathbf{i}_2$, $\mathbf{j}_2$, $\mathbf{k}_2$, par rapport au repère $X_1O_1Y_1Z_1$, sont

$$\left\{\begin{aligned} \mathbf{i}_2&=\alpha_{11}\mathbf{i}_1+\alpha_{12}\mathbf{j}_1+\alpha_{13}\mathbf{k}_1,\\ \mathbf{j}_2&=\alpha_{21}\mathbf{i}_1+\alpha_{22}\mathbf{j}_1+\alpha_{23}\mathbf{k}_1,\\ \mathbf{k}_2&=\alpha_{31}\mathbf{i}_1+\alpha_{32}\mathbf{j}_1+\alpha_{33}\mathbf{k}_1, \end{aligned}\right. \tag{21}$$

où les cosinus directeurs α_{mn} ne dépendent pas du temps t

$$\alpha_{mn}=\text{const.},\quad (m,\,n=1,\,2,\,3), \tag{22}$$

donc

$$\left\{\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{i}_2=\dot{\mathbf{i}}_2=\mathbf{w}_1\times\mathbf{i}_2, && \left(\text{parce que } \frac{\partial^*\mathbf{i}_2}{\partial t}=0\right)\\ &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{j}_2=\dot{\mathbf{j}}_2=\mathbf{w}_1\times\mathbf{j}_2, && \left(\text{parce que } \frac{\partial^*\mathbf{j}_2}{\partial t}=0\right)\\ &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{k}_2=\dot{\mathbf{k}}_2=\mathbf{w}_1\times\mathbf{k}_2, && \left(\text{parce que } \frac{\partial^*\mathbf{k}_2}{\partial t}=0\right) \end{aligned}\right. \tag{23}$$

ce qui implique

$$\begin{aligned} w_{x_2}&=\dot{\mathbf{j}}_2\,.\,\mathbf{k}_2=(\mathbf{w}_1\times\mathbf{j}_2)\,.\,\mathbf{k}_2=\mathbf{k}_2\,.\,(\mathbf{w}_1\times\mathbf{j}_2)=\mathbf{w}_1\,.\,(\mathbf{j}_2\times\mathbf{k}_2)=\mathbf{w}_1\,.\,\mathbf{i}_2,\\ w_{y_2}&=\dot{\mathbf{k}}_2\,.\,\mathbf{i}_2=(\mathbf{w}_1\times\mathbf{k}_2)\,.\,\mathbf{i}_2=\mathbf{i}_2\,.\,(\mathbf{w}_1\times\mathbf{k}_2)=\mathbf{w}_1\,.\,(\mathbf{k}_2\times\mathbf{i}_2)=\mathbf{w}_1\,.\,\mathbf{j}_2,\\ w_{z_2}&=\dot{\mathbf{i}}_2\,.\,\mathbf{j}_2=(\mathbf{w}_1\times\mathbf{i}_2)\,.\,\mathbf{j}_2=\mathbf{j}_2\,.\,(\mathbf{w}_1\times\mathbf{i}_2)=\mathbf{w}_1\,.\,(\mathbf{i}_2\times\mathbf{j}_2)=\mathbf{w}_1\,.\,\mathbf{k}_2 \end{aligned} \tag{24}$$

et alors la relation

(25) $$\mathbf{w}_2 = w_{x_2}\mathbf{i}_2 + w_{y_2}\mathbf{j}_2 + w_z\mathbf{k}_2$$

et les relations (24) impliquent à leur tour

(26) $$\mathbf{w}_2 = (\mathbf{w}_1 \cdot \mathbf{i}_2)\mathbf{i}_2 + (\mathbf{w}_1 \cdot \mathbf{j}_2)\mathbf{j}_2 + (\mathbf{w}_1 \cdot \mathbf{k}_2)\mathbf{k}_2,$$

tandis que l'expression analytique du vecteur $\mathbf{w}_1$, par rapport au repère $X_2O_2Y_2Z_2$, est aussi

(27) $$\mathbf{w}_1 = (\mathbf{w}_1 \cdot \mathbf{i}_2)\mathbf{i}_2 + (\mathbf{w}_1 \cdot \mathbf{j}_2)\mathbf{j}_2 + (\mathbf{w}_1 \cdot \mathbf{k}_2)\mathbf{k}_2,$$

les relations (26) et (27) impliquant donc, évidemment,

(28) $$\mathbf{w}_1 = \mathbf{w}_2, \qquad \text{(q.e.d.)}.$$

Nous voulons maintenantex aminer la position des deux axes instantanés de rotation, des repères $X_1O_1Y_1Z_1$ et $X_2O_2Y_2Z_2$, en sachant jusqu'ici (par les alinéas (a) et (b) de plus haut) seulement que les deux axes ont la même direction. Considérons donc (v. les équations vectorielles des axes instantanés de rotation rapportées au repère fixe, c'est à dire les relations (4) et (5)) le vecteur

(29) $$\mathbf{D} = \left(\mathbf{r}_{O_2} + \frac{\mathbf{w}_2 \times \mathbf{v}_{O_2}}{\mathbf{w}_2^2}\right) - \left(\mathbf{r}_{O_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{\mathbf{w}_1^2}\right),$$

soumis à la condition (20) (ou (28)), c'est à dire

(30) $$\mathbf{D} = \mathbf{r}_{O_2} - \mathbf{r}_{O_1} + \frac{\mathbf{w}_1}{\mathbf{w}_1^2} \times (\mathbf{v}_{O_2} - \mathbf{v}_{O_1}).$$

En substituant la différence $\mathbf{v}_{O_2} - \mathbf{v}_{O_1}$ par l'expression (15), on a

(31) $$\mathbf{D} = \overrightarrow{O_1O_2} + \frac{\mathbf{w}_1}{\mathbf{w}_1^2} \times (\mathbf{w}_1 \times \overrightarrow{O_1O_2}),$$

ou

(32) $$\mathbf{D} = \overrightarrow{O_1O_2} + \mathbf{w}_1\left(\frac{\mathbf{w}_1}{\mathbf{w}_1^2} \cdot \overrightarrow{O_1O_2}\right) - \overrightarrow{O_1O_2},$$

donc

(33) $$\mathbf{D} = \mathbf{w}_1\left(\frac{\mathbf{w}_1}{\mathbf{w}_1^2} \cdot \overrightarrow{O_1O_2}\right)$$

et alors (v. (29))

(34) $$\mathbf{r}_{O_2} + \frac{\mathbf{w}_2 \times \mathbf{v}_{O_2}}{\mathbf{w}_2^2} = \mathbf{r}_{O_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{\mathbf{w}_1^2} + \mathbf{w}_1\left(\frac{\mathbf{w}_1}{\mathbf{w}_1^2} \cdot \overrightarrow{O_1O_2}\right).$$

Donc l'équation (5) de l'axe instantané de rotation du repère $X_2O_2Y_2Z_2$ (c'est à dire l'équation vectorielle de l'axe instantané de rotation du corps rigide, écrite par l'utilisation du repère mobile $X_2O_2Y_2Z_2$, cartésien triorthogonal, solidaire du corps rigide) peut être écrite aussi (v. aussi (20), (28))

(35) $$\mathbf{r} = \mathbf{r}_{O_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{\mathbf{w}_1^2} + \left(\frac{\mathbf{w}_1}{\mathbf{w}_1^2} \cdot \overrightarrow{O_1O_2} + \lambda\right)\mathbf{w}_1,$$

cette équation représentant en même temps l'équation vectorielle de l'axe instantané de rotation du repère $X_1O_1Y_1Z_1$ (c'est à dire l'équation vectorielle de l'axe instantané de rotation du corps rigide, écrite par l'utilisation du repère mobile $X_1O_1Y_1Z_1$, cartésien triorthogonal, solidaire du corps rigide) — v. l'équation vectorielle (4) — seulement l'origine antérieure des valeurs du paramètre λ, sur cet axe instantané de rotation, ayant changée sa position.

En conclusion, nous avons démontré le suivant

Théorème. *Lorsque les repères $X_1O_1Y_1Z_1$ et $X_2O_2Y_2Z_2$, cartésiens triorthogonaux, sont solidaires l'un de l'autre, par exemple lorsque les deux repères sont solidaires d'un même corps rigide, alors les axes instantanés de rotation de ces deux repères coïncident, donc l'axe instantané de rotation de l'un des repères $X_1O_1Y_1Z_1$ ou $X_2O_2Y_2Z_2$, reste invariant pendant la substitution de ce repère par l'autre.*

Reçue le 22 Novembre, 1984

Institut Polytechnique, Jassy,
Chaire de Mécanique

BIBLIOGRAPHIE

1. Vîlcovici V., Bălan Şt., Voinea R., *Mecanică teoretică*. Ed. tehn., Buc., 1968.
2. Onicescu O., *Mecanica*. Ed. tehn., Buc., 1969.
3. Mangeron D., Irimiciuc N., *Mecanica rigidelor cu aplicaţii în inginerie*. Vol. I, Ed. tehn., Buc., 1978.
4. Acker C., *Elemente de mecanică tehnică. Introducere în mecanica rigidului*. Inst. polit. Iaşi, 1982.

ASUPRA INVARIANŢEI AXEI INSTANTANEE DE ROTAŢIE A RIGIDULUI

I. Cazul a două repere carteziene triortogonale solidarizate cu rigidul

(Rezumat)

Considerînd axa instantanee de rotaţie a unui reper cartezian triortogonal definită ca loc geometric al punctelor solidarizate cu reperul, pentru care la un moment dat vectorii viteză sînt paraleli cu vectorul viteză unghiulară al reperului, se demonstrează că dacă două repere carteziene triortogonale oarecare sînt solidare, atunci axele instantanee de rotaţie respective coincid (rezultatul fiind un caz particular al celui ce face obiectul părţii a 2-a a lucrării).

Reperele carteziene triortogonale la care ne referim sînt formate în exclusivitate din vectori unitari.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 531.1

ON THE INSTANTANEOUS ANGULAR VELOCITY VECTOR

BY

ALFRED BRAIER and DANIEL CONDURACHE

1. As we know, the instantaneous angular velocity vector plays an essential rôle in describing the motion of rigid bodies, being involved primarily in the velocity field. In order to introduce this vector, many proceedings are usualy used and presented in the various exposures of the kinematics of rigid bodies.

The purpose of this Note is to point out a simple and direct way leading easily to the angular velocity vector, through an elementary vectorial handling, which reflects clearily the rigidity of the moving object.

2. Let $(\mathcal{C})$ be a rigid body, considered in an arbitrary position at some instant t, and $A, B, C \in (\mathcal{C})$ three arbitrary points, but not lying on the same right line. During the motion $\vec{AB}=\vec{AB}(t)$ and $\vec{AC}=\vec{AC}(t)$ ($\vec{AB}$ means the position vector of B in respect to A, i.e. $\vec{AB}=\vec{OB}-\vec{OA}$, O being the origin of a fixed frame in the tridimensional space).

We express the rigidity of $(\mathcal{C})$ at the level of points A, B, C by

$$\vec{AB}^2=\text{const.}, \qquad \vec{AC}^2=\text{const.}, \qquad \vec{AB}\,.\,\vec{AC}=\text{const.} \tag{1}$$

Differentiation with respect to t (indicated by a dot) gives

$$2\vec{AB}\,.\,\dot{\vec{AB}}=0, \qquad 2\vec{AC}\,.\,\dot{\vec{AC}}=0, \qquad \dot{\vec{AB}}\,.\,\vec{AC}+\vec{AB}\,.\,\dot{\vec{AC}}=0. \tag{2}$$

If at instant t, $\vec{AB}\neq 0$ and $\vec{AC}\neq 0$ then obviously $\dot{\vec{AB}}\perp\vec{AB}$, $\dot{\vec{AC}}\perp\vec{AC}$.

Let now express otherwise this perpendicularity, namely

$$\dot{\vec{AB}}=\vec{\omega}_1\times\vec{AB}, \qquad \dot{\vec{AC}}=\vec{\omega}_2\times\vec{AC}, \tag{3}$$

where $\vec{\omega}_1$, $\vec{\omega}_2$ are two yet unknown vectors, which obviously can not be independent. Indeed, with (3), the last consequence (2) leads immediately to

$$(\vec{\omega}_1-\vec{\omega}_2)\,.\,(\vec{AB}\times\vec{AC})=0, \tag{4}$$

with $\vec{AB}\times\vec{AC}\neq 0$. Therefore, in the most general case, $\vec{\omega}_1-\vec{\omega}_2$ is perpendicular to $\vec{AB}\times\vec{AC}$, that is lies in the plane of triangle ABC, i.e.

(5) $$\vec{\omega}_1-\vec{\omega}_2=\lambda\vec{AB}+\mu\vec{AC}, \qquad \lambda, \mu \in \mathbb{R}.$$

i) If we write $\vec{\omega}_1=\vec{\omega}_2+\lambda\vec{AB}+\mu\vec{AC}$ then (3) become $\dot{\vec{AB}}=(\vec{\omega}_2+\mu\vec{AC})\times\times\vec{AB}$, $\dot{\vec{AC}}=\vec{\omega}_2\times\vec{AC}[=(\vec{\omega}_2+\mu\vec{AC})\times\vec{AC}]$.

ii) If we write $\vec{\omega}_2=\vec{\omega}_1-\lambda\vec{AB}-\mu\vec{AC}$ then (3) become $\dot{\vec{AC}}=(\vec{\omega}_1--\lambda\vec{AB})\times\vec{AC}$, $\dot{\vec{AB}}=\vec{\omega}_1\times\vec{AB}[=(\vec{\omega}_1-\lambda\vec{AB})\times\vec{AB}]$.

We conclude that the single vectorial quantily which matters is (see (5))

(6) $$\vec{\omega}_1-\lambda\vec{AB}=\vec{\omega}_2+\mu\vec{AC}\overset{\text{not}}{=}\vec{\omega}$$

(denoted now by $\vec{\omega}$) and (3) give

(7) $$\dot{\vec{AB}}=\vec{\omega}\times\vec{AB}, \qquad \dot{\vec{AC}}=\vec{\omega}\times\vec{AC}.$$

A routine calculation shows that under (7),

(8) $$\text{if} \quad \vec{AD}=\vec{AB}\times\vec{AC}, \qquad \text{then} \quad \dot{\vec{AD}}=\vec{\omega}\times\vec{AD}.$$

Let now be $M\in(\mathcal{C})$ an arbitrary point. Expressed in basis $\{\vec{AB}, \vec{AC}, \vec{AD}\}$, $\vec{AM}=a\vec{AB}+b\vec{AC}+c\vec{AD}$ where a, b, c are constant quantities during the motion of $(\mathcal{C})$. Using (7) and (8) we get the very general result

(9) $$\dot{\vec{AM}}=\vec{\omega}\times\vec{AM},$$

with the same $\vec{\omega}$ for all $\vec{AM}$.

3. If we know the motions of points A, B, C in respect to a fixed frame, that is we have the vectorial functions $\vec{AB}=\vec{AB}(t)$ and $\vec{AC}=\vec{AC}(t)$ (satisfying the restrictions (1) and supposed to be differentiable) we can effectively get $\dot{\vec{AB}}$ and $\dot{\vec{AC}}$. Then the vectorial product of the two Eqs. (7) with $\vec{\omega}$ as unknown, result in

$$\dot{\vec{AB}}\times\dot{\vec{AC}}=(\vec{\omega}\times\vec{AB})\times(\vec{\omega}\times\vec{AC})=\vec{\omega}[\vec{AC}\,.\,(\vec{\omega}\times\vec{AB})]-\vec{AC}[\vec{\omega}\,.\,(\vec{\omega}\times\vec{AB})].$$

We obtain for the angular velocity

(10) $$\vec{\omega}=\frac{\dot{\vec{AB}}\times\dot{\vec{AC}}}{\dot{\vec{AB}}\,.\,\dot{\vec{AC}}}, \qquad \dot{\vec{AB}}\,.\,\dot{\vec{AC}}\neq 0.$$

Let underline that $\dot{\vec{AB}}\,.\,\dot{\vec{AC}}=0$ means also $\vec{AB}\,.\,\dot{\vec{AC}}=0$ and with (7) $(\vec{\omega}\times\vec{AB})\,.\,\dot{\vec{AC}}=\vec{\omega}\,.\,(\vec{AB}\times\dot{\vec{AC}})=0$ i.e. $\vec{\omega}$ lies in the plane ABC,

(11) $$\vec{\omega}=\alpha\vec{AB}+\beta\vec{AC}, \qquad \alpha, \beta \in \mathbb{R}.$$

Also from (7) and (11) $\dot{\vec{AB}}=\beta\vec{AC}\times\vec{AB}$, $\dot{\vec{AC}}=\alpha\vec{AB}\times\vec{AC}$ and finally

(12) $$\vec{\omega}=(\dot{\vec{AC}}\,.\,\vec{AN})\vec{AB}-(\dot{\vec{AB}}\,.\,\vec{AN})\vec{AC}, \qquad \dot{\vec{AB}}\,.\,\dot{\vec{AC}}=0,$$

where

$$\vec{AN}=\frac{\vec{AB}\times\vec{AC}}{|\vec{AB}\times\vec{AC}|^2}. \tag{13}$$

We get easily that the result (12), (13) hold also in the particular cases $\vec{AB}=\vec{\text{const.}}$ (or $\vec{AC}=\vec{\text{const}}$). Now

$$\vec{\omega}=(\dot{\vec{AC}}\cdot\vec{AN})\vec{AB}, \qquad \vec{AB}=\vec{\text{const.}}, \tag{14}$$

i.e $\vec{\omega}$ is colinear with $\vec{AB}$; it is the case of a rigid body rotating about a fixed axis. If both $\vec{AB}$ and $\vec{AC}$ are constant, then $\vec{\omega}=0$.

4. *Remarks.* 1. If $\vec{AB}=\vec{i}$, $\vec{AC}=\vec{j}$ and $\vec{AD}=\vec{k}$ form a unit orthogonal triad of vectors lying on three rectangular axis fixed on the moving rigid body, then (7) and (8) become the well known Poisson formulae and (10) results in

$$\vec{\omega}=\frac{\dot{\vec{i}}\times\dot{\vec{j}}}{\dot{\vec{i}}\,.\,\dot{\vec{j}}}, \qquad \dot{\vec{i}}\,.\,\dot{\vec{j}}\neq 0, \tag{15}$$

or other two similar formulae.

2. Relation (9) expresses the velocity field of $(\mathcal{C})$: let denote $\vec{OA}\overset{\text{not}}{=}\vec{r}_A$, $\vec{OM}\overset{\text{not}}{=}\vec{r}$, $\vec{v}_A=\dot{\vec{r}}_A$, $\vec{v}=\dot{\vec{r}}$ the respective velocities and $\vec{AM}=\vec{v}-\vec{v}_A$, then

$$\vec{v}=\vec{v}_A+\vec{\omega}\times\vec{AM}. \tag{16}$$

Since $\dot{\vec{AB}}=\vec{v}_B-\vec{v}_A$ and $\dot{\vec{AC}}=\vec{v}_C-\vec{v}_A$, it is obvious that $\vec{\omega}$ exists, is unique and different from zero in each instant when the velocities of the three noncolinear points are not the same. If they are, the rigid performs an instantaneous translation.

3. The independence of $\vec{\omega}$ from the frame fixed in the rigid body can be argued also directly. Let $A', B', C'\in(\mathcal{C})$ be an other set of three noncolinear points. Having in mind (7) we may write

$$\dot{\vec{A'B'}}=\vec{\omega}'\times\vec{A'B'} \quad \text{and} \quad \dot{\vec{A'C'}}=\vec{\omega}'\times\vec{A'C'},$$

with possible an other vector $\vec{\omega}'$. Applying the result (9) to the points A', B', C' we get easily

$$\dot{\vec{A'B'}}=\vec{\omega}\times\vec{A'B'} \quad \text{and} \quad \dot{\vec{A'C'}}=\vec{\omega}\times\vec{A'C'},$$

i.e. $\vec{\omega}-\vec{\omega}'$ must be colinear with both $\vec{A'B'}$ and $\vec{A'C'}$, hence necessarily $\vec{\omega}'=\vec{\omega}$.

Received December 21, 1984

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ASUPRA VECTORULUI VITEZĂ UNGHIULARĂ INSTANTANEE

(Rezumat)

Se indică un mod simplu şi direct de a introduce vectorul viteză unghiulară instantanee în mişcarea rigidului.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 539.43 ; 517.949

ELASTO-STATIC STRESSES IN PLANE STRAIN STATE OF A CIRCULAR CRANKSHIFT BY BOUNDARY ELEMENT METHOD

BY

VICTOR FLORIN POTERAŞU, NICU MIHALACHE and GH. IONESCU

1. Introduction. In the design of machines tools is necessary to know exactly the stress state and the shape of the section during their working. In most cases the strain shape of the section determines the working of the tools in good conditions. The Boundary Elements Method, as a numerical method, allows a study of stress and strain state of the tools with complex section for which the relations known from strength of materials and the elasticity theory cannot be applied. The paper will present briefly the principle of the method and examples to solve some tools of the machines. There will be pointed out some advantages of the proposed method in comparison with other numerical methods.

2. The Boundary Elements Method in Elastostatic. For elements with complex geometrical shape and acted by static or dynamic forces, the stress and strain state is difficult to be determined by means of the relations from strength of materials or from elasticity theory.

In these cases, the numerical methods give solutions very near to the actual results. At the Boundary Elements Method basis are the general equations of the continuum medium of strain and Betti's theorems that are applied to the problem that is to be solved. The method allows the determination of the unknown stresses and displacements in the interior points of the domain if we know a function (the stresses function) and its coefficients of the second order continuous on the boundary of the domain. Using Green's second formula we can pass from the solution of a volume integral to the solution of a surface integral that simplifies numerically the calculus. Thus we can pass from the division of the volume to the surface division for three-dimensions elements and from the surface division to the boundary division for plane elements [2], [3], [7] (Fig. 1).

For finite elastic solids acted by static forces we use the carthesian coordinates system and the standard tensorial notations. Writing the integral equation of a point on the boundary it has the form [4], [7]

$$c_i u^i + \int_S [T_i^{(j)}(P,Q)][u_i^{(j)}(P,Q)]\mathrm{d}s = \int_S [U_i^{(j)}(P,Q)]\,[t_i^{(j)}(P,Q)]\mathrm{d}s + \int_\Omega [U_i^{(j)}(P,Q)][b_i^{(j)}]\,\mathrm{d}\Omega. \tag{1}$$

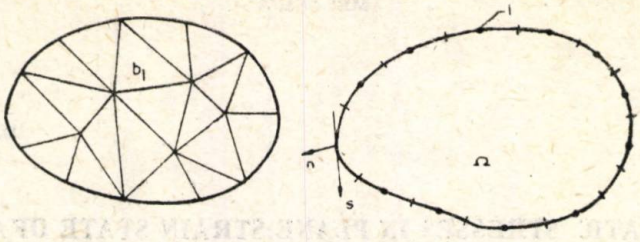

Fig. 1 — Domain discretization; *a*) surface, *b*) boundary.

The fundamental solution $u_i^{(j)}(P, Q)$ are the components of the displacements vector x_i in a point P of the elastic solid, when a concentrated unitary force is applied in the point Q in the direction x_j. Its expression for two-dimensional solids (plane strains) [7] is

$$U_i^{(j)}(P, Q) = \frac{1}{8\pi\mu} \frac{3-4\nu}{1-\nu} \ln\frac{1}{R} \delta_{ij} + \frac{1}{8\pi\mu} \frac{1}{1-\mu} \frac{\partial R}{\partial x_i} \frac{\partial R}{\partial x_j}, \tag{2}$$

where μ—transversal elasticity modulus, ν—Poisson's coefficient, δ_{ij}—delta Kronecker, R—distance between the points P and Q. The component of tractions vector $T_i^{(j)}$ in a point P is given under the form

$$\begin{aligned} T_i^{(j)}(P, Q) = -\frac{1}{4\pi(1-\nu)R} \Bigg\{ \frac{\partial R}{\partial n} \left[(1-2\nu)\delta_{ij} + 2 \frac{\partial R}{\partial x_i} \frac{\partial R}{\partial x_j} \right] - \\ -(1-2\nu) \left[\frac{\partial R}{\partial x_i} n_j - \frac{\partial R}{\partial x_j} n_i \right] \Bigg\}. \end{aligned} \tag{3}$$

If we suppose a simple convex elastic solid Ω in the value problem on the boundary the displacements $u_i(P)$ with $(P \in \Omega)$ must satisfy the equation

$$\mu u_{i,jj}(P) + \frac{\mu}{1-2\nu} u_{j,ij}(P) + b_i(P) = 0 \tag{4}$$

with the conditions on the boundary of the form

$$\begin{aligned} &[u_i(P)]^- = f_i(P) \quad \text{with} \quad (P \in S_\Omega) \\ &[t_i^{(n)}(P)]^- = [C_{ijkl} n_j(P) u_{k,l}(P)]^- = p_i(P), \end{aligned} \tag{5}$$

where $[\ ]^-$ indicates the limit value from the domain interior Ω, the body forces are $b_i(P)$, the displacements of a point on the boundary are $f_i(P)$ and the surface stresses intensity is $p_i(P)$. If we introduce the function $v_i(P)$ expressed in the form

$$\begin{aligned} {}_i(P) = \int_S \psi_j(Q) U_i^{(j)}(P,Q) \mathrm{d}s(Q) - \int_S \varphi_j(Q) T_i^{(j)}(P, Q) \mathrm{d}s(Q) + \\ + \int_\Omega b_i(Q) U_i^{(j)}(P, Q) \mathrm{d}\Omega(Q), \end{aligned} \tag{6}$$

with

$$\varphi_i(P)=f_i(P)\quad (P\in S_d)\qquad \psi_i(P)=p_i(P)\quad (P\in S_t)\qquad S=S_d\cup S_t. \tag{7}$$

If the point $(P\in\Omega^c)$ belongs to the boundary we can write

$$v_i(P)=0 \quad \text{for } (P\in\Omega^c-\text{the infinite domain exterior to } \Omega). \tag{8}$$

The condition equation (8) that is satisfied for a point on the boundary is equivalent with the relation (1) on the boundary and gives the following expressions :

$$u_i(P)=v_i(P)\quad (P\in\Omega)\qquad [u_i(P)]^-=\varphi_i(P),\quad [t_i^{(n)}(P)]^-=\psi_i(P)\quad (P\in S), \tag{9}$$

where C_{ijkl} from the relation (4) represents the elastic constants tensor defined in the form

$$C_{ijkl}=\frac{2\nu}{1-2\nu}\mu\delta_{ij}\delta_{kl}+\mu\delta_{ik}\delta_{jl}+\mu\delta_{il}\delta_{jk}. \tag{10}$$

If the solution $u_i(P)$ of the elasticity equations (4), (5) is known, and the functions $\varphi_i(P)$ and $\psi_i(P)$ of the equation (9), using Somigliana's identities [2], [7] the equation (8) becomes

$$\begin{aligned}&\int_S [t_j^{(n)}(Q)]^- U_i^{(j)}(P,Q)\mathrm{d}s(Q)-\int_S [u_j(Q)]^- T_i^{(j)}(P,Q)\mathrm{d}s(Q)+\\&+\int_\Omega b_j(Q)U_i^j(P,Q)\mathrm{d}\Omega(Q)=\begin{cases} u_i(P) & (P\in\Omega)\\ \frac{1}{2}u_i(P) & (P\in S)\\ 0 & (P\in\Omega^c)\end{cases}\end{aligned} \tag{11}$$

and it represents Eq. (1) written by means of Somigliana's identities with the possibility of applying these relations on computers. If we take into consideration the problem of the circular disc acted by normal forces (the plane strain problem) (Fig. 2) the homogeneous equation becomes

$$\int_\Omega \psi_{0j}(Q)\frac{1}{8\pi\mu}\left[\frac{3-4\nu}{1-\nu}\ln\frac{1}{R}+\frac{1}{1-\nu}\frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_j}\right]\mathrm{d}s(Q)=0, \tag{12}$$

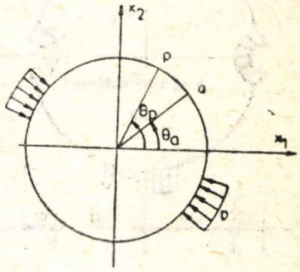

Fig. 2 – Circular disc acted by distributed loads.

where in conformity with Fig. 2 we can write

$$\begin{aligned}&R=2r_0(\theta_P-\theta_Q)\ ;\quad \frac{\partial R}{\partial x_1}=R_{,1}=-\operatorname{sgn}(\theta_P-\theta_Q)\sin\tfrac{1}{2}(\theta_P+\theta_Q),\\&\frac{\partial R}{\partial x_2}=R_{,2}=\operatorname{sgn}(\theta_P-\theta_Q)\cos\tfrac{1}{2}(\theta_P+\theta_Q).\end{aligned} \tag{13}$$

If we represent the functions $\psi_{0i}(P)$ and $\psi_{0j}(Q)$ as a Fourier series of the form

$$\psi_{0i}(P)=\tfrac{1}{2}a_{0i}+\sum_{n=1}^{\infty}\frac{1}{n}\,[a_{ni}\cos n\theta_P+a'_{ni}\sin n\theta_P] \tag{14}$$

substituing in the equation (12) and introducing we get

$$\begin{aligned}&\frac{1}{8\mu}\,\frac{r_0}{1-\nu}\Bigg\{(3-4\nu)\Bigg[a_{0i}\ln\frac{1}{r_0}+\sum_{n=2}^{\infty}\frac{1}{n}\,(a_{ni}\cos n\theta_P+a'_{ni}\sin n\theta_P)\Bigg]+\\ &+\tfrac{1}{2}[a_{0i}+(-1)^i(a_{1i}\cos\theta_P-a'_{1i}\sin\theta_P)-\delta_{1i}(a_{12}\sin\theta_P+a'_{12}\cos\theta_P)-\\ &\delta_{2i}(a_{11}\sin\theta_p+a'_{11}\cos\theta_P)]\Bigg\}=0,\end{aligned} \tag{15}$$

with $0\leqslant\theta_0\leqslant 2\pi$.

Dividing the element in N segments and writing for each segment the expression (11) with $(P\in S)$ there results a system of N equations that being solved gives the solutions on the boundary. To determine the stressses and the displacements in the interior points the relation (11) with $(P\in\Omega)$ is used.

3. Numerical Example. It was studied the stress state in a crankshift having the dimensions, the section shape and loadings being shown in Fig. 3.

Being a problem of plane strain state we take into consideration a disc of unitary thickness acted by forces uniformly distributed (Fig. 3). The division of the contour by menas of Boundary Elements Method is shown in Fig. 4.

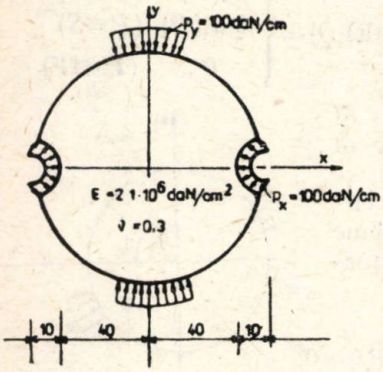

Fig. 3

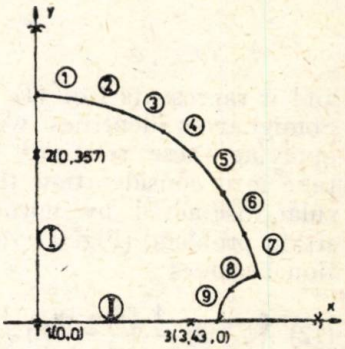

Fig. 4 — Discretization of the disc by BEM.

As the element presents symetrically to the system of axis xOy the division was done only a quarter of the disc. The computer program gives the possibility of the analysis of the values of the displacements on the contour and of the stresses and displacements in the interior points [4]. There were calculated the stresses and the displacements on the coordinates segments given (Fig. 4) in static conditions. The normal stresses values (σ_x, σ_y) are shown in Fig. 5.

If in the elasticity theory the solution for the whole disc acted by concentrated forces is known, the disc with the shape and the loadings given

(Fig. 3) can not be solved by means of known relations. It is worth interesting especially the strained shape of the disc under the act of the forces. It can be studied in Fig. 6. The displacement values are given in Table 1.

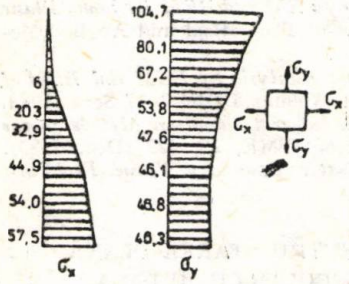

Fig. 5 — Stress variation on the segment *1*.

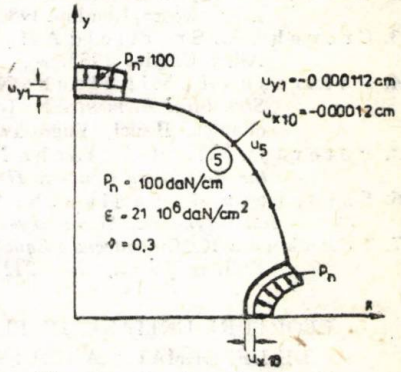

Fig. 6 — Strain of the disc.

Table 1

Knot	ΔX mm	ΔY mm
1	−1.996	−0.358
2	−1.7	−1.133
3	−1.222	−1.378
4	−0.626	−2.093
[illegible]	−0.015	−2.147
6	0.511	−1.877
7	0.883	−1.285
8	1.095	−0.422

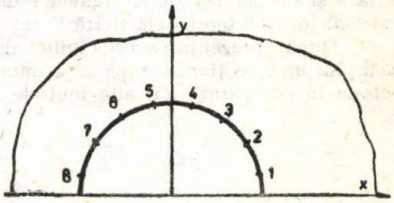

Fig. 7

The solution by means of Finite Elements Method is possible but it requires a far greater input data, a higher computer time. The computer time by means of Boundary Elements Method is 24 minutes and it is 47% smaller than the one used by means of Finite Elements Method. Great values of the stresses in the points on the boundary can be observed. Using the boundary conditions we can notice that the stresses σ_y on the contour calculated numerically by menas of Boundary Elements Method give a 4.7% error.

4. Conclusions. The Boundary Elements Method gives the possibility to solve technical-engineering problems numerically in static conditions. The division of the domain only on the boundary reduces the size of the algebrical equations system. The precision of the method is dermined by the methods that are used to solve the integrals. In this method the technique of weighted residual and Gauss's quadrature formulas are used. In a future paper to be we shall present the numerical solution in the elastodynamic domain of the engineering problems.

Received September 20, 1984

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REFERENCES

1. Brebbia C.A., Ferante A.J., *Computational Methods for the Solutions of Engineering Problems.* Pentech Press, London, 1978.
2. Brebbia C.A., Walker S., *Boundary Element Technique in Engineering.* Newness—Butterworth, London, 1980.
3. Crouch S.L., Starfield A.M., *Boundary Element Methods in Solid Mechanics.* George Allen Unwin, 1983.
4. Poteraşu V.F., Mihalache N., *Comparison Study by FEM and BEM for Some Plane Structural Elements.* XVI-th Yugoslav Congress of Theoretical and Applied Mechanics, Becici, Yugoslavia, June 1984.
5. Poteraşu V.F., Mihalache N., *Static and Dynamic Analysis by FEM and BEM of the Reinforced Concrete Hybrid Structures.* IKM Weimar, DDR, 2—7 Sept. 1984.
6. Shibahara M., Taniguchi Y., *Application of the Integral Equation Method to the Elastodynamic Boundary-Value Problems.* Bull of JSME, **26**, 222 (Dec. 1983).
7. Takakuda K., *On Integral Equation Methods for Elastic Boundary Value Problems.* Bull. of JSME, **26**, 212 (Febr. 1983).

EFORTURI UNITARE ÎN ELASTO-STATICĂ PENTRU STAREA PLANĂ DE DEFORMAŢIE A UNUI ARBORE COTIT CIRCULAR CU METODA ELEMENTELOR DE FRONTIERĂ

(Rezumat)

Metoda elementelor de frontieră, ca metodă numerică, permite studiul stării de eforturi unitare şi de deformaţie a organelor de maşini cu secţiune complexă pentru care rezistenţa materialelor şi teoria elasticităţii nu pot fi aplicate.

După prezentarea principiilor de bază ale metodei elementelor de forntieră se studiază exemplul unei secţiuni complexe a unui arbore circular cotit. Se subliniază avantajele acestei metode în comparaţie cu alte metode numerice.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 539.3

THE GREEN'S FUNCTIONS FOR SHORT TIME IN THE GENERALIZED THEORY OF THERMOELASTICITY

II. IMPULSIVE BODY FORCES AND HEAT SOURCE

BY

ELISABETA RUSU

1. Introduction. The basic equations of the generalized theory of thermoelasticity for an isotropic solid with a center of symmetry at each point are [1]

$$\begin{cases} \mu\Delta\mathbf{u}+(\lambda+\mu)\text{grad div }\mathbf{u}-m\text{ grad}(\theta+\alpha\dot{\theta})+\mathbf{b}=\rho\ddot{\mathbf{u}}, \\ k\Delta\theta-d\dot{\theta}-h\ddot{\theta}-\theta_0 m\text{ div }\dot{\mathbf{u}}=-\rho r. \end{cases} \tag{1.1}$$

We consider the following initial and boundary conditions:

$$\begin{cases} \mathbf{u}(x,0)=0,\quad \dot{\mathbf{u}}(x,0)=0,\quad \theta(x,0)=0,\quad \dot{\theta}(x,0)=0; \\ \lim\limits_{x\to\infty}\mathbf{u}(x,t)=0,\quad \lim\limits_{x\to\infty}\theta(x,t)=0. \end{cases} \tag{1.2}$$

In the first part of this paper [2], a general solution of the dynamic coupled thermoelastic Eqs. (1.1) has been obtained for arbitrary distribution of body forces and heat source in an infinite body, by the use of Laplace-Fourier transform :

$$\begin{cases} u_j^{(2)}(\zeta,p)=\dfrac{1}{\lambda+2\mu}\left[\dfrac{\zeta^2+q+f}{\Delta_1\zeta^2}\zeta_h b_k^{(2)}\zeta_j+\dfrac{m}{\Delta_1}\mathrm{i}\zeta_j(1+\alpha_p)\dfrac{\rho r^{(2)}}{k}\right]+\dfrac{\zeta^2 b_j^{(2)}-\zeta_j\zeta_k b_k^{(2)}}{\mu\zeta^2(\zeta^2+\beta_2^2)}, \\ \theta^{(2)}(\zeta,p)=\dfrac{\theta_0^m}{k(\lambda+2\mu)}\dfrac{p}{\Delta_1}\mathrm{i}\zeta_j b_j^{(2)}+\dfrac{\zeta^2+\beta_1^2}{\Delta_1}\dfrac{\rho r^{(2)}}{k}, \end{cases} \tag{1.3}$$

where

$$\begin{gathered} q=\frac{dp}{k},\quad f=\frac{hp^2}{k},\quad c_1^2=\frac{\lambda+2\mu}{\rho},\quad c_2^2=\frac{\mu}{\rho},\quad \beta_1=\frac{p}{c_1},\quad \beta_2=\frac{p}{c_2}, \\ \varepsilon=\frac{\theta_0 m^2}{d(\lambda+2\mu)},\quad \zeta^2=\zeta_k\zeta_k, \\ \Delta_1=(\zeta^2+q+f)(\zeta^2+\beta_1^2)+\varepsilon q(1+\alpha p)\zeta^2=(\zeta^2-\lambda_1^2)(\zeta^2-\lambda_2^2). \end{gathered} \tag{1.4}$$

In the second part of paper we will derive the solution for an impulsive body force and heat source acting in an infinite medium, i.e. fundamental solution.

2. Impulsive Body Force and Heat Source. In this section we week a solution of the Eqs. (1.1) when an impulsive body force is applied at the origin.

For such à body force we may write

$$b_j^\alpha=\delta(x)\delta(y)\delta(z)\delta(t)\delta_{\alpha j},\quad r=0\ ;\ (j=1,2,3), \tag{2.1}$$

if the body force is parallel to Ox_α axis ($\alpha=1, 2, 3$).

From (1.3) and (2.1) we find, respectively for $\alpha=1, 2, 3$

$$\begin{cases} u_j^{\alpha(2)}(\zeta,p)=\dfrac{1}{\lambda+2\mu}\dfrac{\zeta^2+q+f}{\Delta_1\zeta^2}\zeta_\alpha\zeta_j+\dfrac{\zeta^2\delta_{\alpha j}-\zeta_j\zeta_\alpha}{\mu\zeta^2(\zeta^2+\beta_2^2)}\quad (j=1,2,3),\\ \theta^{\alpha(2)}(\zeta,p)=\dfrac{\theta_0 mp}{k(\lambda+2\mu)\Delta_1}\,i\zeta_\alpha. \end{cases} \tag{2.2}$$

If we consider the case when a heat source is applied impulsively at the origin

$$b_j^4=0,\ (j=1,2,3)\ ;\quad \frac{\rho r^4}{k}=\delta(x)\delta(y)\delta(z)\delta(t) \tag{2.3}$$

then from (1.3) and (2.3) we find

$$u_j^{4(2)}(\zeta,p)=\frac{m}{\rho c_1^2\Delta_1}\,i\zeta_j(1+\alpha p),\quad \theta^{4(2)}(\zeta,p)=\frac{\zeta^2+\beta_1^2}{\Delta_1}. \tag{2.4}$$

On inverting the Fourier transform with respect to $\zeta_1, \zeta_2, \zeta_3$, we obtain from (2.2) ($\alpha=1, 2, 3$)

$$\begin{cases} u_j^{\alpha(1)}(x,p)=-\dfrac{1}{\rho}\dfrac{\partial^2}{\partial x_j\partial x_\alpha}\left(\dfrac{I_1}{c_1^2}-\dfrac{I_2}{c_2^2}\right)+\dfrac{1}{\rho c_2^2}I_3\delta_{j\alpha},\quad (j=1,2,3)\\ \theta^{\alpha(1)}(x,p)=-\dfrac{\theta_0 mp}{\rho k c_1^2}\dfrac{\partial I_4}{\partial x_\alpha} \end{cases} \tag{2.5}$$

and from (2.4)

$$u_j^{4(1)}(x,p)=-\frac{m(1+\alpha P)}{\rho c_1^2}\frac{\partial I_4}{\partial x_j},\quad (j=1,2,3),\quad \theta^{4(1)}(x,p)=I_5, \tag{2.6}$$

where

$$R^2=x_kx_k,\quad \mathcal{F}^{-1}\left(\frac{1}{\zeta^2+k^2}\right)=\frac{1}{4\pi R}e^{-kR},\quad \mathcal{F}^{-1}\left(\frac{1}{\zeta^2}\right)=\frac{1}{4\pi R};$$

$$I_1=\mathcal{F}^{-1}\left(\frac{\zeta^2+q+f}{\zeta^2\Delta_1}\right)=\mathcal{F}^{-1}\left[\frac{q+f}{\zeta^2\lambda_1^2\lambda_2^2}+\frac{\lambda_1^2+q+f}{\lambda_1^2(\lambda_1^2-\lambda_2^2)(\zeta^2-\lambda_1^2)}-\frac{\lambda_2^2+q+f}{\lambda_2^2(\lambda_1^2-\lambda_2^2)(\zeta^2-\lambda_2^2)}\right]=$$

$$=\frac{1}{4\pi R}\left(\frac{q+f}{\lambda_1^2\lambda_2^2}+\frac{\lambda_1^2+q+f}{\lambda_1^2(\lambda_1^2-\lambda_2^2)}e^{-i\lambda_1R}-\frac{\lambda_2^2+q+f}{\lambda_2^2(\lambda_1^2-\lambda_2^2)}e^{-i\lambda_2R}\right),$$

$$I_2=\mathcal{F}^{-1}\left(\frac{1}{\zeta^2(\zeta^2+\beta_2^2)}\right)=\frac{1}{4\pi R\,\beta_2^2}(1-e^{-\beta_2R}),\quad I_3=\mathcal{F}^{-1}\left(\frac{1}{\zeta^2+\beta_2^2}\right)=\frac{1}{4\pi R}e^{-\beta_2R},$$

$$I_4=\mathcal{F}^{-1}\left(\frac{1}{\Delta_1}\right)=\frac{1}{\lambda_1^2-\lambda_2^2}\frac{1}{4\pi R}(e^{-i\lambda_1R}-e^{-i\lambda_2R}), \tag{2.7}$$

$$I_5=\mathcal{F}^{-1}\left(\frac{\zeta^2+\beta_1^2}{\Delta_1}\right)=\frac{1}{4\pi R}\left(\frac{\lambda_1^2+\beta_1^2}{\lambda_1^2-\lambda_2^2}\,e^{-i\lambda_1 R}-\frac{\lambda_2^2+\beta_1^2}{\lambda_1^2-\lambda_2^2}\,e^{-i\lambda_2 R}\right).$$

From (2.5) and (2.7) we get for $\alpha=1, 2, 3$:

$$(2.8)\qquad \begin{cases} u_j^{\alpha(1)}(x,p)=-\dfrac{1}{4\pi\rho}\left(\dfrac{x_j x_\alpha}{R^3}Q-\dfrac{1}{R^2}S\delta_{\alpha j}\right), & j=1, 2, 3, \\ \theta^{\alpha(1)}(x,p)=\dfrac{\theta_0 m p}{4\pi\rho k}\dfrac{x_\alpha}{R^2}\mathcal{P}. \end{cases}$$

From (2.6) and (2.7) we get

$$(2.9)\qquad u_j^{4(1)}(x,p)=\frac{m(1+\alpha p)}{4\pi\rho}\frac{x_j}{R^2}\mathcal{P},\quad \theta^{4(1)}(x,p)=I_5,$$

where

$$\mathcal{P}=\frac{1}{c_1^2(\lambda_1^2-\lambda_2^2)}\left[\left(i\lambda_1+\frac{1}{R}\right)e^{-i\lambda_1 R}-\left(i\lambda_2+\frac{1}{R}\right)e^{-i\lambda_2 R}\right],$$

$$(2.10)\qquad \begin{aligned} Q=&\frac{\lambda_1^2+q+f}{c_1^2\lambda_1^2(\lambda_1^2-\lambda_2^2)}\left[(i\lambda_1)^2+\frac{3i\lambda_1}{R}+\frac{3}{R^2}\right]e^{-i\lambda_1 R}-\frac{\lambda_2^2+q+f}{c_1^2\lambda_2^2(\lambda_1^2-\lambda_2^2)}\left[(i\lambda_2)^2+\right.\\ &\left.+\frac{3i\lambda_2}{R}+\frac{3}{R^2}\right]e^{-i\lambda_2 R}+\frac{1}{c_2^2\beta_2^2}\left(\beta_2^2+\frac{3\beta_2}{R}+\frac{3}{R^2}\right)e^{-\beta_2 R},\\ S=&\frac{\lambda_1^2+q+f}{c_1^2\lambda_1^2(\lambda_1^2-\lambda_2^2)}\left(i\lambda_1+\frac{1}{R}\right)e^{-i\lambda_1 R}-\frac{\lambda_2^2+q+f}{c_1^2\lambda_2^2(\lambda_1^2-\lambda_2^2)}\left(i\lambda_2+\frac{1}{R}\right)e^{-i\lambda_2 R}+\\ &+\frac{1}{c_2^2\beta_2^2}\left(\beta_2+\frac{1}{R}\right)e^{-\beta_2 R}+\frac{1}{c_2^2 R}e^{-\beta_2 R}. \end{aligned}$$

Let us obtain the inverse Laplace transforms for small values of time as in [3], [4].

From Abel's theorem it follows that high values of p correspond to small values of t.

Expandind the expressions of roots λ_1, λ_2 of the Eq. (1.4) $\Delta_1=0$ in Mac Laurin series and retaining the first four terms and disregarding ε^2 in comparison with ε, we obtain

$$(2.11)\qquad \begin{aligned} \lambda_1=&\pm\frac{ip}{c_1}\left[1+\frac{B\varepsilon\alpha}{2(1-A)}+\frac{\varepsilon B(1-A+B\alpha)}{2p(1-A)^2}+\frac{\varepsilon B^2(1-A+B\alpha)}{2p^2(1-A)^3}+\right.\\ &\left.+\frac{\varepsilon B^3(1-A+B\alpha)}{2p^3(1-A)^4}+\dots\right],\\ \lambda_2=&\pm\frac{ip\sqrt{A}}{c_1}\left\{1-\frac{\varepsilon B\alpha}{2(1-A)}+\frac{B}{2Ap}\left[1-\frac{\varepsilon B\alpha}{2(1-A)}-\frac{\varepsilon A(1-A+B\alpha)}{(1-A)^2}\right]-\right.\\ &-\frac{B^2}{2p^2}\left[\frac{1}{4A^2}+\frac{\varepsilon(1-A+B\alpha)}{(1-A)^3}+\frac{\varepsilon(1-A+B\alpha)}{2A(1-A)^2}-\frac{\varepsilon B\alpha}{8A^2(1-A)}\right]+\\ &\left.+\frac{B^2}{2p^3}\left[\frac{1}{8A^3}-\frac{\varepsilon B\alpha}{16A^3(1-A)}-\dots\right]\right\}. \end{aligned}$$

We take

$$e^{-i\lambda_1 R}=e^{-pk_1}e^{-\omega_1(1-A)}\left(1-\frac{B\omega_1}{p}\right),\qquad k_1=\frac{R}{c_1}\left[1+\frac{B\varepsilon\alpha}{2(1-A)}\right],$$

$$e^{-i\lambda_2 R}=e^{-pk_2}e^{-k_3}\left(1-\frac{\varepsilon k_4}{p}\right)\left(1+\frac{\omega_2}{p}+\frac{\omega_2^2}{2p^2}\right),$$

$$k_2=\frac{\sqrt{A}\,R}{c_1}\left[1-\frac{B\varepsilon\alpha}{2(1-A)}\right]$$

(2.12)

$$k_3=\frac{\sqrt{A}BR}{2Ac_1}\left[1-\frac{B\varepsilon\alpha}{2(1-A)}-\frac{\varepsilon A(1-A+B\alpha}{(1-A)^2}\right],$$

$$\omega_1=\frac{R(1-A+B\alpha)}{(1-A)^3 2c_1},$$

$$k_4=\frac{\sqrt{A}RB^2}{2c_1}\left[\frac{B}{8A^2(1-A)}-\frac{1-A+B\alpha}{(1-A)^3}-\frac{1-A+B\alpha}{2A(1-A)^2}\right].$$

We have [1]

$$1-A+B\alpha=1+\frac{c_1^2}{k}(\alpha d-h)>0\,.$$

Assuming that $1-A>0$, it follows that $\omega_1>0$, $k_1>0$, $k_2>0$, $k_3>0$.

From (2.10), (2.11), (2.12) disregarding ε^2 as a small quantity comparated to ε, we obtain

$$\mathcal{P}=-\frac{1}{(1-A)p}\left[\left(P_1+\frac{P_2}{p}+\frac{P_3}{p^2}+\frac{P_4}{p^3}\right)e^{-pk_1}e^{-\omega_1(1-A)}-\right.$$
$$\left.-\left(P_5+\frac{P_6}{p}+\frac{P_7}{p^2}+\frac{P_8}{p^3}+\frac{P_9}{p^4}\right)e^{-pk_2}e^{-k_3}\right],$$

(2.13) $$Q=\left(\sum_{k=0}^{4}Q_k p^{-4}\right)e^{-pk_1}e^{-\omega_1(1-A)}-\left(Q_5+\frac{Q_6}{p}+\frac{Q_7}{p^2}+\frac{Q_8}{p^3}+\frac{Q_9}{p^4}\right)e^{-pk_2}e^{-k_3},$$

$$S=\sum_{k=1}^{4}(S_k p^{-k})e^{-pk_1}e^{-\omega_1(1-A)}+\left(\frac{1}{c_2^2R}+\frac{1}{pc_2}+\frac{1}{Rp^2}\right)e^{-\frac{pR}{c_2}}-$$
$$-\left(\frac{S_5}{p}+\frac{S_6}{p^2}+\frac{S_7}{p^3}+\frac{S_8}{p^4}\right)e^{-pk_2}e^{-k_3},$$

where

$$P_1=\frac{1}{c_1}\left[1-\frac{\varepsilon B\alpha(1+3A)}{2(1-A)^2}\right],$$

$$P_2=\frac{B}{c_1(1-A)}\left[1-\frac{3\varepsilon B\alpha(1+A)}{(1-A)^2}-\frac{\varepsilon(1+A+B\alpha)}{1-A}+\frac{\varepsilon B\alpha}{2(1-A)}+\right.$$
$$\left.+\frac{\varepsilon(1-A+B\alpha)}{2(1-A)}\right]+\frac{1}{R}\left[1-\frac{B\varepsilon\alpha(1+A)}{(1-A)^2}\right]-\frac{B\omega_1}{c_1},$$

$$P_3=\frac{B^2}{c_1(1-A)^2}\left\{1-\varepsilon\left[1+\frac{3(1+A+B\alpha)}{1-A}+\frac{6B\alpha(1+A)}{(1-A)^2}+\frac{B\alpha+1+A+B\alpha}{2(1-A)}\right]+\right.$$

$$+\frac{B}{R(1-A)}\left[1-\frac{3B\varepsilon\alpha(1+A)}{(1-A)^2}-\frac{\varepsilon(1+A+B\alpha)}{1-A}\right]-\frac{B^2\omega_1}{c_1(1-A)}-\frac{B\omega_1}{R},\ldots$$

$$P_5=\frac{\sqrt{A}}{c_1}\left[1-\frac{\varepsilon B\alpha(3+A)}{2(1-A)^2}\right],$$

$$P_6=\frac{\sqrt{A}}{c_1}\left\{\frac{B(1+A)}{2A(1-A)}-\frac{B\varepsilon}{(1-A)^2}\left[\frac{3B\alpha(1+A)}{1-A}+\frac{3-3A+4B\alpha}{2}+\right.\right.$$

$$\left.\left.+\frac{B\alpha(1-A)}{4A}+\frac{B\alpha(1+A)}{4A^2}\right]\right\}+\frac{1}{R}\left[1-\frac{B\varepsilon\alpha(1+A)}{(1-A)^2}\right]+$$

$$+\frac{B^2R}{8Ac_1^2}\left[1-\frac{B\varepsilon\alpha(3+A)}{2(1-A)^2}\right]-k_4\frac{\sqrt{A}}{c_1},$$

$$Q_0=-\frac{1}{c_1^2}\left[1-\frac{AB\varepsilon\alpha}{(1-A)^2}\right],$$

$$Q_1=-\frac{1}{c_1^2}\left\{\frac{3c}{R}\left[1-\frac{\varepsilon B\alpha(1+A)}{2(1-A)^2}\right]+\frac{\varepsilon B(A^2-A-B\alpha-AB\alpha)}{(1-A)^3}\right\}+\frac{B\omega_1}{c_1^2},$$

$$Q_2=-\frac{1}{c_1^2}\left\{\frac{3c_1^2}{R}\left[1-\frac{B\varepsilon\alpha}{(1-A)^2}\right]+\frac{3c_1\varepsilon B(A^2-1-3B\alpha-AB\alpha)}{2(1-A)^3}+\right.$$

$$\left.+\frac{\varepsilon B^2(A^2-1-2B\alpha)}{(1-A)^4}\right\}+\frac{3B}{c_1R}\omega_1,$$

$$Q_3=\frac{3B\omega_1}{R^2}-\frac{1}{c_1^2}\left[\frac{\varepsilon B^3(1-A+B)}{(1-A)^4}+\frac{3c_1\varepsilon B^2(A^2+2A-3+5B\alpha+AB\alpha)}{2R(1-A)^4}-\right.$$

$$\left.-\frac{3c_1^2\varepsilon B(1-A+2B\alpha)}{R^2(1-A)^3}\right],$$

$$Q_5=\frac{\varepsilon AB\alpha}{c_1^2(1-A)^2},\quad Q_6=\frac{B\varepsilon}{c_1^2(1-A)^2}\left(\frac{A-A^2+B\alpha+AB\alpha}{1-A}+\frac{c_1\sqrt{A}\,3\alpha}{R}\right)+Q_5\omega_2\,;$$

$$S_1=-\frac{1}{c_1}\left[1-\frac{B\varepsilon\alpha(1+A)}{2(1-A)^2}\right],$$

$$S_2=-\frac{1}{c_1}\left\{\frac{\varepsilon B(A-1-3B\alpha)}{2(1-A)^2}+\frac{c_1}{R}\left[1-\frac{B\varepsilon\alpha}{(1-A)^2}\right]\right\}-\frac{B\omega_1}{c_1},\ \ldots$$

$$S_5=\frac{\varepsilon B\sqrt{A}\alpha}{c_1(1-A)^2},\quad S_6=\frac{\varepsilon B\sqrt{A}}{c_1(1-A)^2}\left(\frac{1-A+2B\alpha}{1-A}+\frac{B\alpha}{2A}+\frac{c_1\alpha}{R\sqrt{A}}\right),\ldots$$

From (2)...(13), (2.8) and respectively (2.13) and (2.9) we obtain the expressions quite suitable for Laplace inversion. For $\alpha=1$

$(b_1=b_1^1=\delta(x)\delta(y)\delta(z)\delta(t),\ b_2=0,\ b_3=0,\ r=0)$ we get

$$u_1^{1(1)}(x,p)=-\frac{1}{4\pi\rho R^2}\left\{\left[\frac{x_1^2}{R}Q_0+\frac{1}{p}\left(\frac{x_1^2}{R}Q_1-S_1\right)+\frac{1}{p^2}\left(\frac{x_1^2}{R}Q_2-S_2\right)+\right.\right.$$

$$\left.+\frac{1}{p^3}\left(\frac{x_1^2}{R}Q_3-S_3\right)+\frac{1}{p^4}\left(\frac{x_1^2}{R}Q_4-S_4\right)\right]e^{-pk_1}e^{-\omega_1(1-A)}-\left[\frac{x_1^2}{R}Q_5+\right.$$

$$+\frac{1}{p}\left(\frac{x_1^2}{R}Q_6+S_5\right)+\frac{1}{p^2}\left(\frac{x_1^2}{R}Q_7+S_6\right)+\frac{1}{p^3}\left(\frac{x_1^2}{R}Q_8+S_7\right)+$$

$$\left.\left.+\frac{1}{p^4}\left(\frac{x_1^2}{R}Q_9+S_8\right)\right]e^{-pk_2}e^{-k_2}+\left(\frac{1}{c_2^2R}+\frac{1}{c_2p}+\frac{1}{Rp^2}\right)e^{-p\frac{R}{c_2}}\right\},$$

$$u_2^{1(1)}(x,p)=-\frac{1}{4\pi\rho}\frac{x_1x_2}{R^3}\left\{\left(Q_0+\frac{Q_1}{p}+\frac{Q_2}{p^2}+\frac{Q_3}{p^3}+\frac{Q_4}{p^4}\right)e^{-pk_1}e^{-\omega_1(1-A)}-\right.$$

(2.14)

$$\left.-\left(Q_5+\frac{Q_6}{p}+\frac{Q_7}{p^2}+\frac{Q_8}{p^3}+\frac{Q_9}{p^4}\right)e^{-pk_2}e^{-k_2}\right\},$$

$$u_3^{1(1)}(x,p)=-\frac{1}{4\pi\rho}\frac{x_1x_3}{R^3}\left\{\left(Q_0+\frac{Q_1}{p}+\ldots\right)e^{-pk_1}e^{-\omega_1(1-A)}-\right.$$

$$\left.-\left(Q_5+\frac{Q_6}{p}+\ldots\right)e^{-pk_2}e^{-k_2}\right\},$$

$$\theta^{1(1)}(x,p)=-\frac{\theta_0 m}{4\pi\rho A}\frac{x_1}{R^2}\frac{1}{1-A}\left\{\left(P_1+\frac{P_2}{p}+\frac{P_3}{p^2}+\ldots\right)e^{-pk_1}e^{-\omega_1(1-A)}-\right.$$

$$\left.-\left(P_5+\frac{P_6}{p}+\frac{P_7}{p^2}+\ldots\right)e^{-pk_2}e^{-k_2}\right\}.$$

Thus on inverting we obtain

$$u_1^1(x_1,t)=-\frac{1}{4\pi\rho R^2}\left\{e^{-\omega_1(1-A)}\left[\frac{x_1^2Q_0}{R}\delta(t-k_1)+H(t-k_1)\left(\frac{x_1^2Q_1}{R}-S_1\right)+\right.\right.$$

$$\left.+(t-k_1)\left(\frac{x_1^2Q_2}{R}-S_2\right)+\frac{1}{2!}(t-k_1)^2\left(\frac{x_1^2Q_3}{R}-S_3\right)\right]-e^{-k_2}\left[\frac{x_1^2}{R}Q_5\delta(t-k_2)+\right.$$

$$\left.+H(t-k_2)\left(\frac{x_1^2}{R}Q_6+S_5\right)+(t-k_2)\left(\frac{x_1^2}{R}Q_7+S_6\right)+\frac{1}{2!}(t-k_2)^2\left(\frac{x_1^2}{R}Q_8+S_7\right)\right]+$$

$$\left.+\frac{1}{c_2^2R}\delta\left(t-\frac{R}{c_2}\right)+\frac{1}{c_2}H\left(t-\frac{R}{c_2}\right)+\frac{1}{R}\left(t-\frac{R}{c_2}\right)H\left(t-\frac{R}{c_2}\right)\right\},$$

$$u_2^1(x,t)=-\frac{1}{4\pi\rho}\frac{x_1x_2}{R^3}\left\{\left[Q_0\delta(t-k_1)+Q_1H(t-k_1)+Q_2H(t-k_1)\frac{t-k_1}{1!}+\right.\right.$$

$$\left.\left.+\ldots\right]e^{-\omega_1(1-A)}-\left[Q_5\delta(t-k_2)+Q_6H(t-k_2)+Q_7H(t-k_2)\frac{t-k_2}{1!}+\ldots\right]e^{-k_2}\right\},$$

$$u_3^1(x, t) = \frac{x_3}{x_2} u_2^1(x, t),$$

$$\theta^{(1)}(x, t) = -\frac{\theta_0 m x_1}{4\pi\rho k R^2} \frac{1}{(1-A)} \left\{ \left[P_1\delta(t-k_1) + P_2 H(t-k_1) + P_3 H(t-k_1) \frac{t-k_1}{1!} + \right.\right.$$

$$\left.\left. + \ldots \right] e^{-\omega_1(1-A)} - \left[P_5\delta(t-k_2) + P_6 H(t-k_2) + P_7 H(t-k_2) \frac{t-k_2}{1!} + \ldots \right] e^{-k_2} \right\}.$$

In a similar way we can write the solution for $\alpha = 2, 3$ and for the heat source. The solution (2.15) can be used as a Green type solution for determination of the displacement and temperature fields for sufficient small values of the time ; it is the fundamental solution.

Received September 15, 1984

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REFERENCES

1. Green A.E., Lindsay K.A., *Thermoelasticity*. Jour. of Elasticity, **2**, 1–7 (1972).
2. Rusu E., *The Green's Functions, for Short Time, in the Generalized Theory of Thermoelasticity*. (I) *General solution*. Bul. Inst. polit. Iaşi, **XXIX(XXXIII)**, *1–4*, s. I(1983).
3. Hetnarski R.B., *The Fundamental Solution of the Coupled Thermoelastic Problems for Small Time*. Arch. Mech. Stos., **16**, 23–31 (1964).
4. Soos E., *The Green's Function (for Small Time) in the Linear Theory of the Coupled Thermoelasticity*. Arch. Mech. Stos., **18**, *1*, 101 (1966).

FUNCŢII GREEN PENTRU TIMP SCURT ÎN O TEORIE GENERALIZATĂ A TERMOELASTICITĂŢII

II. Forţe impulsive şi sursă impulsivă

(Rezumat)

În partea I-a lucrării [2] s-a obţinut transformata Laplace-Fourier a soluţiei (deplasările şi temperatura) unei probleme cu forţe masice şi sursă oarecare, dată. În partea a II-a se deduce soluţia problemei, cînd forţa masică şi sursa sînt impulsive.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 539.3

ON THE EXISTENCE OF SOLUTIONS IN THE THEORY OF LINEAR THERMOELASTICITY WITH MICROSTRUCTURE

BY

SILVIA GABRIELA CIUMAŞU

1. Introduction. Using some results of [1] we prove a theorem of existence, uniqueness of the solutions in the linear theory of thermoelasticity with microstructure established in [2].

2. Statement of the Problem. We refer the motion of the continuum to a fixed system of rectangular cartesian axes Ox_i $(i=1, 2, 3)$. The basic equations of the linear dynamic theory of microstructure thermoelasticity for homogeneous centrosymmetryc solids, are :

i) *The equations of motion*

$$t_{ji,j}+\rho F_i=\rho\ddot{u}_i, \quad t_{ij}=\sigma_{ij}+\tau_{ij}, \tag{2.1}$$
$$\sigma_{ij}+\mu_{kij,k}+\rho F_{ij}=Y_{js}\ddot{\varphi}_{is}\,;$$

ii) *The equation of energy*

$$\theta_0\rho S+q_{i,i}-\rho r=0\,; \tag{2.2}$$

iii) *The constitutive equations*

$$\tau_{ij}=A_{ijkl}\varepsilon_{kl}+B_{ijkl}\gamma_{kl}+a_{ij}(\theta+\alpha\dot{\theta}),$$
$$\sigma_{ij}=B_{klij}\varepsilon_{kl}+C_{ijkl}\gamma_{kl}+c_{ij}(\theta+\alpha\dot{\theta}), \tag{2.3}$$
$$\mu_{kij}=D_{kijlsm}\varkappa_{lsm}, \quad q_i=-\theta_0 k_{ij}\theta_{,j},$$
$$\rho S=a+d\theta+h\dot{\theta}-a_{ij}\varepsilon_{ij}-c_{ij}\gamma_{ij}\,;$$

iv) *The geometrical equations*

$$\varepsilon_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i}), \quad \gamma_{ij}=u_{j,i}-\varphi_{ij}, \quad \varkappa_{ijk}=\varphi_{jk,i}. \tag{2.4}$$

From the residual *entropy inequality* [2] we have the condition

$$(\alpha d-h)\dot{\theta}^2+k_{ij}\theta_{,i}\theta_{,j}\geqslant 0. \tag{2.5}$$

We add the conditions [3]

$$\alpha d-h\geqslant 0, \quad \rho>0, \quad \theta_0>0, \quad \alpha>0, \quad h>0, \quad k_{ij}\theta_{,i}\theta_{,j}\geqslant 0.$$

The coefficients in (2.3) have the following symmetries :

$$
\begin{aligned}
&A_{ijkl}=A_{klij}=A_{jikl}, \quad C_{ijkl}=C_{klij}, \quad Y_{is}=Y_{si}, \\
&B_{ijkl}=B_{jikl}, \quad D_{kijlsm}=D_{lsmkij}, \quad k_{ij}=k_{ji}.
\end{aligned} \tag{2.6}
$$

We have used the notations from [2]. Introducing (2.3) in (2.1), (2.2) we obtain the following system :

$$
\begin{aligned}
&[(A_{ijkl}+B_{klij})\varepsilon_{kl}+(B_{ijkl}+C_{ijkl})\gamma_{kl}+(a_{ij}+c_{ij})(\theta+\alpha\dot{\theta})]_{,i}+\rho F_j=\rho\ddot{u}_j, \\
&B_{klij}\varepsilon_{kl}+C_{ijkl}\gamma_{kl}+C_{ij}(\theta+\alpha\dot{\theta})+D_{kijlsm}\varkappa_{lsm,k}+\rho F_{ij}=Y_{js}\ddot{\varphi}_{is}, \\
&d\dot{\theta}+h\ddot{\theta}-a_{ij}\dot{\varepsilon}_{ij}-c_{ij}\dot{\gamma}_{ij}-k_{ij}\theta_{,ji}=\frac{1}{\theta_0}\rho r.
\end{aligned} \tag{2.7}
$$

Let V be an open, bounded, connected set in the Euclidean space E_3 properly regular in the sense of F i c h e r a [1]. By ∂V we denote the boundary of V; let $t_0>0$. To the equations (2.1)...(2.4) we add the boundary conditions

$$
u_i=0, \quad \varphi_{ij}=0, \quad \theta=0, \quad \text{on} \quad \partial V\times[0, t_0) \tag{2.8}
$$

and initial conditions :

$$
\begin{aligned}
\chi&=\{u_i(x, 0), \varphi_{ij}(x, 0), \dot{u}_i(x, 0), \dot{\varphi}_{ij}(x, 0), \theta(x, 0), \dot{\theta}(x, 0)\}= \\
&=\{u_{i0}(x), \varphi_{ij0}(x), v_{i0}(x), v_{ij0}(x), T_0(x), \dot{T}_0(x)\}, \text{ on } \overline{V}.
\end{aligned} \tag{2.9}
$$

Here $u_{i0}(x)$, $\varphi_{ij0}(x)$, $v_{i0}(x)$, $v_{ij0}(x)$, $T_0(x)$, $\dot{T}_0(x)$ are given functions on $\overline{V}$. By a solution of the first boundary value problem of the microstructure thermoelasticity in the cylinder $Q_{t_0}=V\times(0, t_0)$ we mean an ordered array $(\mu_i, \varphi_{ij}, \theta)$ which satisfies system (2.7) for all $(x, t)\in Q_{t_0}$, the boundary conditions (2.8) and the initial conditions (2.9).

3. Theorem of Existence and Uniqueness of Solutions. By $C^m(\overline{V})$ we denote the class of scalar fields possessing derivatives up to the m-th order in V which are continuous on $\overline{V}$. By $\mathcal{C}^m(\overline{V})$ we denote the class of vector fields with twelve components in $C^m(\overline{V})$.

By $W_m(V)$ respective $\mathcal{W}(V)$ we denote the Hilbert space obtained as the completion of $C^m(\overline{V})$ respectively $\mathcal{C}^m(\overline{V})$ by means of the norm $\|\cdot\|_{W_m(V)}$, respectively $\|\cdot\|_{\mathcal{W}_m(V)}$ induced by the inner product

$$
(\varphi, \psi)_{W_m(V)}=\sum_{k=0}^{m}\int_V \varphi_{,i_1\ldots i_k}\cdot\psi_{,i_1\ldots i_k}dV,
$$

$$
(\bar{v}, \bar{w})_{\mathcal{W}_m(V)}=\sum_{j=1}^{3}(v_j, \omega_j)_{W_m(V)}.
$$

We will use as norms in Cartesian product of the normed spaces the sum of the norms of the factor spaces. Let H be a Banach space and $[0, t_0)$ atime interval. We define by $C^m([0, t_0)\,;\,H)$ the set of functions from $[0, t_0)$

to H, which possess on $[0, t_0)$ time derivatives in H up to the m-th order, continuous on $[0, t_0)$. We introduce analogously the symbols $L_1((0, t_0) ; H)$, $L_2((0, t_0) ; H)$.

We denote

$\hat{C}^1(V)=\{R\in C^1(V) : R=0 \text{ on } \partial V\}$;

$\hat{\mathcal{C}}'(V)=\{u\equiv(u_i, \varphi_{ij})\in C^1(\bar{V}) : u_i=0, \varphi_{ij}=0, \text{ on } \partial V\}$;

$\hat{W}_1(V)\equiv$ the completion of $\hat{C}^1(V)$ by means of $\|\cdot\|_{W_1(V)}$;

$\hat{\mathcal{W}}_1(V)\equiv$ the completion of $\hat{\mathcal{C}}^1(V)$ by means of $\|\cdot\|_{\mathcal{W}_1(V)}$.

Let $\theta, R\in C^1(V), u=(u_i, \varphi_{ij}), v=(v_i, \psi_{ij})\in\hat{\mathcal{C}}^1(V)$.

We define

$$\Phi(\theta, R)=\int_V k_{ij}R_{,i}\theta_{,j}\mathrm{d}V, \tag{3.1}$$

$$\Psi(u, v)=\frac{1}{2}\int_V [A_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(v)+C_{ijkl}\gamma_{ij}(u)\gamma_{kl}(v)+ \tag{3.2}$$

$$+D_{kijlsm}\varkappa_{kij}(u)\varkappa_{lsm}(v)+B_{ijkl}\gamma_{ij}(u)\varepsilon_{kl}(v)+\gamma_{ij}(v)\varepsilon_{kl}(u)]\mathrm{d}V.$$

Obviously, Φ and Ψ may be extended by continuity onto $\hat{W}_1(V)$ and $\hat{\mathcal{W}}_1(V)$ respectively. In order to establish the existence theorems we need the following additional assumptions :

i) there exists positive constants N_1, N_2 such that

$$\Phi(R, R)\geqslant N_1\int_V R_{,i}R_{,j}\mathrm{d}V \quad \forall R\in\hat{C}^1(V) ; \tag{3.3}$$

$$\Psi(v, v)\geqslant N_2\int_V \sum_{i,j,k=1}^{3} [\varepsilon_{ij}^2(v)+\gamma_{ij}^2(v)+\varkappa_{ijk}^2(v)\mathrm{d}V, \quad \forall v\in\hat{\mathcal{C}}^1(V) ; \tag{3.4}$$

ii) there exists a constant I, $I>0$ so that

$$Y_{jk}\xi_{ij}\xi_{ik}\geqslant I\xi_{ij}\xi_{ij}, \quad \forall\xi_{ij}. \tag{3.5}$$

We observe that (3.3), (3.4) can be extended onto $\hat{W}_1(V)$, $\hat{\mathcal{W}}_1(V)$ respectively. For $v\in\hat{\mathcal{W}}(V)$, there exists a constant k, such that we have [4]

$$\Psi(v, v)\geqslant k\int_V (v_iv_i+\psi_{ij}\psi_{ij}+v_{i,j}v_{i,j}+\psi_{ik,j}\psi_{ik,j})\,\mathrm{d}V. \tag{3.6}$$

Let $t_0>0$. We consider the sets

$\tilde{C}^0(V)=\{(F, r) : F=(F_i, F_{ij})\in\mathcal{C}^0(V), \quad r\in C^0(\bar{V})\}$;

$\tilde{C}^1(V)=\{(u, \theta) : u=(u_i, \varphi_{ij}), \quad (u, \theta)\in\hat{\mathcal{C}}(V)\times\hat{C}^1(V)\}$;

$\mathcal{F}(Q_{t_0})=C^\infty([0, t_0) ; \tilde{C}^1(V))$;

$\mathring{\mathcal{F}}(Q_{t_0})=\{(v, R) : (v, R)\in\mathcal{F}(Q_{t_0}) \text{ and } v=0 \text{ on } V\times 0\}$.

Let $y=(u, \theta)\in\mathcal{F}(Q_{t_0})$, $\delta=(v, R)\in\dot{\mathcal{F}}(Q_{t_0})$, $z=(F, r)\in C^{\infty}([0, t_0)\,;\,\tilde{C}^0(V))$,

$$\chi=(u_0, \dot{u}_0, T_0, \dot{T}_0) \text{ with } (u_0, T_0)\in\tilde{\mathcal{C}}^1(V),\ \dot{u}_0\in\hat{\mathcal{C}}^1(V),\ \dot{T}_0\in\hat{C}^1(V).$$

Let $V(Q_{t_0})$ be the Hilbert space obtained as the completion of $\mathcal{F}(Q_{t_0})$ by means of the norm $|\,.\,|$ induced by the inner product

$$<(u, \theta), (v, R)>=\int_0^{t_0}\int_V\Big[u_iv_i+\varphi_{ij}\psi_{ij}+\dot{u}_i\dot{v}_i+\dot{\varphi}_{ij}\dot{\psi}_{ij}+u_{i,j}v_{i,j}+$$

$$+\varphi_{ij,k}\psi_{ij,k}+\theta R+(\theta+\dot{\theta})(R+\dot{R})+\theta_{,i}R_{,i}+\int_0^t\dot{\theta}R\,d\tau+\int_0^t\theta_{,i}R_{,i}\,d\tau\Big]dV\,dt.$$

Let $\mathring{V}(Q_{t_0})$ be the closed linear manifold of $V(Q_{t_0})$ obtained as the completion of $\mathring{\mathcal{F}}(Q_{t_0})$ by means of $|\,.\,|$. Let $\mathring{U}(Q_{t_0})$ be the Hilbert space obtained as the completion of $\mathring{\mathcal{F}}(Q_{t_0})$ by means of the norm induced by the inner product

$$[y, \delta]=<y, \delta>+<\dot{y}, \dot{\delta}>.$$

Let $G(V)$ be the completion of $\tilde{C}^0(V)$ by means of the norm $\|\,.\,\|_{\mathcal{V}(V)\times W(V)}$. Let $H_0(V)$ be the completion of the set $\tilde{C}^1(V)\times\tilde{C}^1(V)$ by means of the norm

$$|(u_i, w_i, \varphi_{ij}, \xi_{ij}, \theta, \tau)|_0=\tfrac{1}{2}\int_V\Big[\rho w_iw_i+Y_{ij}\xi_{ik}\xi_{jk}+A_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(u)+$$

$$+2B_{ijkl}\varepsilon_{ji}(u)\gamma_{kl}(u)+C_{ijkl}\gamma_{ij}(u)\gamma_{kl}(u)+D_{kijlsm}\varkappa_{kij}(u)\varkappa_{lsm}(u)+\alpha k_{ij}\theta_{,i}\theta_{,j}+$$

$$+\alpha(\alpha d-h)\left(\frac{\theta}{\alpha}\right)^2+\frac{h}{\alpha}(\theta+\alpha\tau)^2\Big]\,dV.$$

D e f i n i t i o n. $y=(u, \theta)\in V(Q_{t_0})$ will be called a finite energy solution of (2.7), (2.8) in Q_{t_0} with initial conditions $\chi=(u_0, \dot{u}_0, T_0, \dot{T}_0)\in H_0(V)$ and source $z=(F, r)\in L_1((0, t_0)\,;\,G(V))$, if y satisfies the following conditions:

$$L(y, \delta)=D(z, \delta)+E(\chi, \delta),\ \forall\delta\in U(Q_{t_0})\,;\quad u(t)\xrightarrow{W_0(V)}u_0,\ t\to 0, \tag{3.7}$$

where

$$L(y, \delta)=\int_V\int_0^{t_0}\Big\{(t-t_0)\,[-t_{ji}\dot{v}_{i,j}+\sigma_{ij}\dot{\varphi}_{ij}-\mu_{kij}\dot{\varphi}_{ij,k}+\rho\dot{u}_i\ddot{v}_i+Y_{js}\dot{\psi}_{is}\ddot{\varphi}_{ij}+$$

$$+(d\theta+h\dot{\theta}-a_{ij}\varepsilon_{ij}-c_{ij}\gamma_{ij})(\dot{R}+\alpha\ddot{R})+\alpha k_{ij}\dot{\theta}_{,j}R_{,i}]+\rho\dot{u}_i\dot{v}_i+Y_{js}\dot{\psi}_{is}\dot{\varphi}_{ij}+$$

$$+(d\theta+h\dot{\theta}-a_{ij}\varepsilon_{ij}-c_{ij}\gamma_{ij})(R+\alpha\dot{R})+\alpha k_{ij}\theta_{,j}R_{,i}+\int_0^t k_{ij}\theta_{,j}R_{,i}\,d\tau\Big\}dV\,dt,$$

$$D(z,\delta)=-\int_V\int_0^{t_0}(t-t_0)\left[\rho F_i\dot{v}_i+\rho F_{ij}\dot{\varphi}_{ij}+\frac{\rho r(R+\alpha\dot{R})}{T_0}\right]\mathrm{d}V\mathrm{d}t,$$

$$E(\chi,\delta)=t_0\int_V\Big[(dT_0+h\dot{T}_0-a_{ij}\dot{\varepsilon}_{0ij}-c_{ij}\dot{\gamma}_{0ij})(R+\alpha\dot{R})_{t=0}+\rho\dot{u}_{i0}\dot{v}_i\,|_{t=0}+$$

$$+Y_{js}\dot{\varphi}_{is0}\dot{\psi}_{ij}|_{t=0}\Big]\mathrm{d}V.$$

It is easy to verify the identity

$$L(\delta,\delta)=\tfrac{1}{2}\int_0^{t_0}\int_V\Big[(\rho\dot{v}_i\dot{v}_i+Y_{js}\dot{\psi}_{is}\dot{\psi}_{ij}+A_{ijkl}\varepsilon_{ij}\varepsilon_{kl}+2B_{ijkl}\gamma_{kl}\varepsilon_{ij}+$$

$$+C_{ijkl}\gamma_{kl}\gamma_{ij}+D_{kijlsm}+\varkappa_{kij}\varkappa_{lsm}+\alpha(\alpha d-h)\left(\frac{R}{\alpha}\right)^2+\frac{h}{\alpha}(R+\alpha\dot{R})^2+$$

$$+2(\alpha d-h)\int_0^t\dot{R}^2\mathrm{d}\tau+\alpha k_{ij}R_{,i}R_{,j}+2\int_0^t k_{ij}R_{,j}R_{,i}\mathrm{d}\tau\Big]\mathrm{d}V\mathrm{d}t+ \tag{3.8}$$

$$+\tfrac{1}{2}t_0\int_V[\rho\dot{v}_i\dot{v}_i+Y_{js}\dot{\psi}_{is}\dot{\psi}_{ij}+\alpha h\dot{R}^2]_{t=0}\mathrm{d}V.$$

Using Schwarz's inequality and Sobolev's embedding theorem one can easily see that $L(y,\delta)$, $D(z,\delta)$ can be extended by continuity onto $V(Q_{t_0})\times\times\mathring{U}(Q_{t_0})$, $L_1((0,t_0);\ G(V))\times\mathring{U}(Q_{t_0})$ respectively. $E(\chi,\delta)$ makes sense for $\chi\in H_0(V)$, $\delta\in\mathring{U}(Q_{t_0})$. Using (3.3)...(3.5) and (3.8) it follows that there exists a constant $c_1>0$ such that

$$|\delta|^2\leqslant c_1L(\delta,\delta). \tag{3.9}$$

Following D a f e r m o s [1] we can prove the following results.

L e m m a. *Let $y\in V(Q_{t_0})$ be a finite energy solution of* (2.7), (2.8), (2.9) *in Q_{t_0}, with initial conditions $\chi\equiv(u_0,\dot{u}_0,T_0,\dot{T}_0)$ and source $z=(f_i,F_{ij},r)$. Then*

$$\tilde{y}(x,t)\equiv\int_0^t y(x,\tau)\mathrm{d}\tau$$

is also a finite energy solution of (2.7), 2.8) *in Q_{t_0}, with initial conditions $\tilde{\chi}=(0,u_0,0,T_0)$ and source $\tilde{z}=(\tilde{f}_i,\tilde{F}_{ij},\tilde{r})$, where*

$$\rho\tilde{F}_i=\int_0^t\rho F_i\mathrm{d}\tau+\rho\dot{u}_{0i}-\alpha(a_{ij}+c_{ij})T_{0,i},$$

$$\rho F_{ij} = \int_0^t \rho F_{ij} d\tau + Y_{js}\dot{\varphi}_{0is} - \alpha c_{ij} T_0,$$

$$\tilde{\rho r} = \frac{1}{T_0}\int_0^t \rho r d\tau + d T_0 + h \dot{T}_0 - a_{ij}\varepsilon_{0ij} + c_{ij}\gamma_{0ij}.$$

Theorem 1. *There exists at most one finite energy solution of* (2.7), (2.8) *in* Q_{t_0} *with given initial conditions and given source.*

Theorem 2. *There exists a finite energy solution* $y \equiv (u, \theta) \in \mathring{V}(Q_{t_0})$ *of* (2.7), (2.8) *in* Q_{t_0} *with zero initial conditions and source* $z \equiv (F, r) \in L_2((0, t_0); G(V))$. *There exists a constant* $c > 0$, *depending on* c_1 *of* (3.9) *such that*

$$|y| \leqslant c \, \| z \|_{L_2(Q_{t_0}) \times L_2(Q_{t_0})}. \tag{3.10}$$

Received February 4, 1983

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REFERENCES

1. Dafermos C.M., *On the Existence and the Asymptotic Stability of Solutions to the Equations of Linear Thermoelasticity.* Arch. Rat. Mech. Anal., **29**, 241 (1968).
2. Neagu S.G., *A Generalized Theory of Microstructure in Linear Thermoelasticity.* Bul. Inst. polit. Iaşi, **XXVI(XXX)**, *3–4*, s. IV (1980).
3. Ciumaşu S.G., *A Uniqueness Theorem for the Theory of Linear Thermoelasticity with Microstructure.* Bul. Inst. polit. Iaşi, **XXVIII (XXXII)**, *1–4*, s. I (1982).
4. Hlavaček I., Hlavaček M., *On the Existence and Uniqueness of Solution and some Variational Principles in Linear Theories of Elasticity with Couple-stresses.* Aplicace Matematiky, **14** (1969).

TEOREMĂ DE EXISTENŢĂ A SOLUŢIILOR ÎN TEORIA GENERALIZATĂ A TERMOELASTICITĂŢII CU MICROSTRUCTURĂ

(Rezumat)

Se demonstrează teorema de existenţă a soluţiei ecuaţiilor teoriei generalizate a termoelasticităţii cu microstructură liniare, pentru un mediu omogen, anizotrop, cu centru de simetrie, pentru prima problemă la limită.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 539.3

THE VIBRATION OF CIRCULAR PLATE

BY

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1. Let be a circular plate having a radius d, with the middle surface in the Ox_1x_2 plane and we denote by h his thickness. In the Ox_1x_2 plane the polar-coordinates are used. At the initial moment $t=0$, the plate is loaded with a constant and uniform load P distributed over the faces of plate $x_3=\pm h/2$. The plate is supported at the edge $r=d$ so that the bending moment is

$$M_r(d)=N\frac{1-\nu}{d}\left.\frac{\mathrm{d}w}{\mathrm{d}r}\right|_{r=d}, \quad \text{with} \quad N=\frac{Eh^3}{12(1-\nu^2)}, \tag{1.1}$$

where E is Young's module, ν is Poisson ratio, w is the deflection of plate.

It is known [5] that the moment in a plate is

$$M_r=N\left(\frac{\mathrm{d}^2w}{\mathrm{d}r^2}+\frac{\nu}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right). \tag{1.2}$$

From (1.1) and (1.2) it results that on edge $r=d$ we have $\nabla^2 w=0$, where ∇^2 is Hamilton's operator in two dimension.

The boundary conditions are

$$w(d,t)=0, \quad \nabla^2 w(d,t)=0, \quad \theta(d,t)=0\,; \tag{1.3}$$

and the initial conditions are

$$w(r,0)=0, \quad \dot{w}(r,0)=0\,, \quad \theta(r,0)=0, \quad \dot{\theta}(r,0)=0. \tag{1.4}$$

2. In two Notes [3] and [4] we have deduced the governing equations for a homogeneous, isotropic, linear thermoelastic plate in a generalized theory of thermoelasticity proposed by I. Müller [1], A. E. Green and K. A. Lindsay [2]. Using the Laplace transform

$$W(r,p)=\int_0^\infty w(r,t)\mathrm{e}^{-pt}\mathrm{d}t, \qquad T(r,p)=\int_0^\infty \tau(r,t)\mathrm{e}^{-pt}\mathrm{d}t, \tag{2.1}$$

with

$$\tau(r,t)=\frac{12}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\theta x_3\,\mathrm{d}x_3, \tag{2.2}$$

we deduced the following equations

$$\nabla^4 W + C_1 \nabla^2 T + C_2 W = \bar{B}, \qquad \nabla^2 W + C_4 T + C_5 \nabla^2 T = \bar{H}, \tag{2.3}$$

where

$$\bar{B} = \int_0^\infty \frac{B}{N} e^{-pt} dt + \frac{\rho h}{N}\left[\dot{w}(r, 0) + p w(r, 0) + \frac{\alpha\beta(1+\nu)}{3} \nabla^2 T(r, 0)\right],$$

$$C_1 = \frac{a(1+\nu)}{3}(1-\alpha p), \quad C_2 = \frac{\rho h}{N} p^2,$$

$$C_4 = \frac{3k(1-\nu)}{E\theta_0 p \beta}\left\{-\frac{p}{\varkappa}\left[1+\frac{\varepsilon}{2(1-\nu)}\right] - \frac{p^2}{\varkappa}\left[\alpha^* + \frac{\varepsilon\alpha}{2(1-\nu)}\right] - \frac{12}{h^2}\right\},$$

$$C_5 = \frac{3k(1-\nu)}{E\beta\theta_0 p},$$

$$\bar{H} = \frac{3k(1-\nu)}{E\beta\theta_0 p}\left\{\int_0^\infty -\frac{H}{k} e^{-pt} dt + \nabla^2 w(r, 0) - \frac{1}{\varkappa}\left[1+\frac{\varepsilon}{2(1-\nu)}\right] - \right. \tag{2.4}$$

$$\left. -\frac{1}{\varkappa}\left[\alpha^* + \frac{\varepsilon\alpha}{2(1-\nu)}\right][\dot{\tau}(r, 0) + p\tau(r, 0)]\right\},$$

$$B = s_{\sigma.\sigma} + P, \quad H = \rho R - \frac{6(\bar{p}+\bar{q})}{h^2}, \quad s_\gamma = [t_{3\gamma} x_3]_{-\frac{h}{2}}^{\frac{h}{2}} + \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} F_\gamma x_3 dx_3,$$

$$P = [t_{33}]_{-\frac{h}{2}}^{\frac{h}{2}} + \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} F_3 dx_3, \quad \bar{p} = q_3\left(r, -\frac{h}{2}\right), \quad \bar{q} = q_3\left(r, \frac{h}{2}\right),$$

$$Q_\sigma = \frac{12}{h^3}\int_{-\frac{h}{2}}^{\frac{h}{2}} q_\sigma x_3 dx_3, \quad R = \frac{12}{h^3}\int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{r} x_3 dx_3.$$

Here, t_{ij} are the components of stress tensor ; ρ is density ; F_i—the components of body forces ; q is the amount of heat generated in a unit volume per unit time ; $\bar{r}$ is externally applied heat per unit mass ; θ_0 is the constant reference temperature ; β, k, α, α^* are constants ; a superposed dot denotes the differentiation with respect to time t ; Latin indices have the range 1, 2, 3, and Greek indices take the value 1 and 2 only ; ε is a small constant.

Using the same method as E. Inan [6] we introduce the function

$$U = \nabla^2 W + C_1 T, \tag{2.5}$$

and the system of equations (2.3) can be written as

$$\nabla^2 W + C_1 T - U = 0, \quad \nabla^2 W + C_4 T + C_5 \nabla^2 T = \bar{H}, \quad \nabla^2 U + C_2 W = \bar{B}. \tag{2.6}$$

Using the multiplicators η, μ, ν we find the equation

$$\nabla^2[(\eta+\mu)W + \mu C_5 T + \nu U] + [\nu C_2 W + (\eta C_1 + \mu C_4)T - \eta U] = \mu\bar{H} + \nu\bar{B}. \tag{2.7}$$

We take unknown function η, μ, ν so as to have

$$M(\eta+\mu) = \nu C_2, \quad M\mu C_5 = \eta C_1 + \mu C_4, \quad M\nu = -\eta, \tag{2.8}$$

where M verifies the following equation

$$(M^2 + C_2)(MC_5 - C_4) + M^2 C_1 = 0. \tag{2.9}$$

At each root M_i of this equation we can write (2.7) as

$$\nabla^2 V_i + M_i V_i = f_i, \tag{2.10}$$

where

$$V_i = (\eta_i + \mu_i)W + \mu_i C_5 T + \nu_i U, \quad f_i = \mu_i \bar{H} + \nu_i \bar{B}, \tag{2.11}$$

and η_i, μ_i, ν_i are the solution of algebraic Eqs. (2.8) for $M = M_i$:

$$\eta_i = M_i(C_4 - C_5 M_i), \quad \mu_i = -C_i M_i, \quad \nu_i = -(C_4 - C_5 M_i). \tag{2.12}$$

3. Now, we consider a circular plate of radius d and thickness h, for which the boundary and initial conditions are (1.3) and (1.4). We assume that the component q_3 of heat flux is identicaly zero, the body forces F_i and the externaly applied heat $\bar{r}$ are neglected. Under these assumptions, from (2.4) we have

$$B = P, \quad H = 0, \quad \bar{H} = 0, \quad \bar{B} = \frac{P}{Np}. \tag{3.1}$$

Using $(2.11)_2$ and (2.12) we find

$$f_i = \nu_i \bar{B} = -(C_4 - C_5 M_i)\frac{P}{Np}. \tag{3.2}$$

It is necessary to calculate the roots of Eq. (2.9). For this purpose we observe that the coefficients of Eq. (2.9) can be writen in the form

$$\frac{C_1 - C_4}{C_5} = \bar{\nu}p^2 + \bar{\alpha}p + \bar{\beta}, \quad C_2 = \bar{\gamma}p^2, \quad -\frac{C_4 C_2}{C_5} = \bar{\psi}p^4 + \bar{\theta}p^2 + \bar{\delta}p^3, \tag{3.3}$$

where

$$\begin{gathered} \bar{\nu} = \frac{1}{\varkappa}(\alpha^* + \varepsilon\alpha), \quad \bar{\alpha} = \frac{1}{\varkappa}(1+\varepsilon), \quad \bar{\beta} = \frac{12}{h^2}, \quad \bar{\gamma} = \frac{\rho h}{N}, \\ \bar{\psi} = \frac{\rho h}{N\varkappa}\left[\alpha^* + \frac{\varepsilon\alpha}{2(1-\nu)}\right], \quad \bar{\delta} = \frac{\rho h}{N\varkappa}\left[1 + \frac{\varepsilon}{2(1-\nu)}\right], \quad \bar{\theta} = \frac{12\rho h}{Nk^2}. \end{gathered} \tag{3.4}$$

If we divide the polynomial $M^3 + \frac{C_1 - C_4}{C_5} M^2 + C_2 M - \frac{C_2 C_4}{C_5}$ by $M + (\bar{\nu}p^2 + \bar{\alpha}p + \bar{\beta})$, then, the rest of division is of same order as ε which

is a dimensionless factor of order 10^{-4} [6] and in consequence we can take an approximate value for the first root of Eq. (2.9) as

$$M_1 = -(\bar{\nu}p^2 + \bar{\alpha}p + \bar{\beta}). \tag{3.5}$$

For M_2 and M_3 we have the values

$$M_2 = i\sqrt{\bar{\gamma}}\,p, \quad M_3 = -i\sqrt{\bar{\gamma}}\,p, \quad (i = \sqrt{-1}). \tag{3.6}$$

Using (2.4), (3.3), (3.5) and (3.6), from (3.2) we have

$$f_1 = -\frac{P\beta(1+\nu)}{3Np}(1+\alpha p),$$

$$f_2 = -\frac{3Pk(1-\nu)}{EN\beta\theta_0}\left\{-\frac{1}{\varkappa}\left[1+\frac{\varepsilon}{2(1-\nu)}\right] - \frac{i\sqrt{\bar{\gamma}}}{p} - \frac{p}{\varkappa}\left[\alpha^* + \frac{\varepsilon\alpha}{2(1-\nu)}\right] - \frac{\bar{\beta}}{p}\right\}, \tag{3.7}$$

$$f_3 = -\frac{3Pk(1-\nu)}{EN\beta\theta_0 p}\left\{-\frac{1}{\varkappa}\left[1+\frac{\varepsilon}{2(1-\nu)}\right] + \frac{i\sqrt{\bar{\gamma}}}{p} - \frac{p}{\varkappa}\left[\alpha^* + \frac{\varepsilon\alpha}{2(1-\nu)}\right] - \frac{\bar{\beta}}{p}\right\},$$

where

$$\varepsilon = \frac{2\beta^2\varkappa E}{9k}\theta_0\frac{1+\nu}{1-2\nu} \quad \text{and} \quad \frac{1}{\varkappa} = \frac{\rho c}{k}.$$

4. The Eq. (2.10) can be written in another form if we take into account the expression of Hamilton's operator in polar-coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{4.1}$$

Because the deformation is symmetric about Ox_3 axis, it results that the deflection w and the function τ depend on r and t, only. Then the V_i are functions of r and p.

In this case, Eq. (2.10) can be written in the form

$$\frac{d^2V_k}{dr^2} + \frac{1}{r}\frac{dV_k}{dr} + M_kV_k = f_k, \quad (k = 1, 2, 3). \tag{4.2}$$

The solution of Bessel's equation (4.2) is

$$V_k = A_k J_0(\sqrt{M_k}r) + \frac{f_k}{M_k}, \tag{4.3}$$

where J_0 is Bessel's function. Using the boundary conditions (1.3), from (2.5) and (2.11) we have

$$V_k(d, p) = 0. \tag{4.4}$$

Substituting (4.4) into (4.3) we determine the constants of integration

$$A_k = -\frac{f_k}{M_k J_0(\sqrt{M_k}d)}. \tag{4.5}$$

Using (2.11), (2.12), (2.4), (4.3) and (4.5) we find

$$W(r,p)=\frac{1}{C_5}\left\{\frac{\frac{f_1}{M_1}\left[1-\frac{J_0(\sqrt{M_1}\,r)}{J_0(\sqrt{M_1}\,d)}\right]}{[\bar{\nu}p^2+(\bar{\alpha}+i\sqrt{\bar{\gamma}})p+\bar{\beta}][\bar{\nu}p^2+(\bar{\alpha}-i\sqrt{\bar{\gamma}})p+\bar{\beta}]}+\right.$$

$$\left.+\frac{\frac{f_2}{M_2}\left[1-\frac{J_0(\sqrt{M_2}\,r)}{J_0(\sqrt{M_2}d)}\right]}{2i\sqrt{\bar{\gamma}}p[\bar{\nu}p^2+(\bar{\alpha}+i\sqrt{\bar{\gamma}})p+\bar{\beta}]}+\frac{\frac{f_3}{M_3}\left[1-\frac{J_0(\sqrt{M_3}\,r)}{J_0(\sqrt{M_3}\,d)}\right]}{2i\sqrt{\bar{\gamma}}p[\bar{\nu}p^2+(\bar{\alpha}-i\sqrt{\bar{\gamma}})p+\bar{\beta}]}\right\}. \tag{4.6}$$

5. To apply an inverse Laplace transform we shall define

$$g_{11}=\frac{1+\alpha p}{(\bar{\nu}p^2+\bar{\alpha}p+\bar{\beta})[\bar{\nu}p^2+(\bar{\alpha}+i\sqrt{\bar{\gamma}})p+\bar{\beta}][\bar{\nu}p^2+(\bar{\alpha}-i\sqrt{\bar{\gamma}})p+\bar{\beta}]}, \tag{5.1}$$

$$g_{21}=\frac{\frac{p}{\varkappa}\left(1+\frac{\varepsilon}{2(1-\nu)}\right)+i\sqrt{\bar{\gamma}}\,p+\frac{p^2}{\varkappa}\left(\alpha^*+\frac{\varepsilon\alpha}{2(1-\nu)}\right)+\bar{\beta}}{p^3[\bar{\nu}p^2+(\bar{\alpha}+i\sqrt{\bar{\gamma}})p+\bar{\beta}]}, \tag{5.2}$$

$$g_{31}=\frac{\frac{p}{\varkappa}\left(1+\frac{\varepsilon}{2(1-\nu)}\right)-i\sqrt{\bar{\gamma}}\,p+\frac{p^2}{\varkappa}\left(\alpha^*+\frac{\varepsilon\alpha}{2(1-\nu)}\right)+\bar{\beta}}{p^3[\bar{\nu}p^2+(\bar{\alpha}-i\sqrt{\bar{\gamma}})p+\bar{\beta}]}, \tag{5.3}$$

$$g_{12}=-g_{11}\frac{J_0(\sqrt{M_1}r)}{J_0(\sqrt{M_1}d)},\quad g_{22}=-g_{21}\frac{J_0(\sqrt{M_2}r)}{J_0(\sqrt{M_2}d)},\quad g_{32}=-g_{31}\frac{J_0(\sqrt{M_3}r)}{J_0(\sqrt{M_3}d)}, \tag{5.4}$$

so that the W can be written as

$$W(r,p)=\mathcal{A}(g_{11}-g_{12})+\mathcal{B}(g_{21}-g_{22}+g_{31}-g_{32}), \tag{5.5}$$

with

$$\mathcal{A}=\frac{E\beta^2\theta_0(1+\nu)P}{9k(1-\nu)N},\quad \mathcal{B}=-\frac{P}{2\bar{\gamma}N}. \tag{5.6}$$

Applying the inverse Laplace transform, from (5.5) we have

$$w(r,t)=\mathcal{L}^{-1}W(r,p)=\mathcal{A}(\mathcal{L}^{-1}g_{11}-\mathcal{L}^{-1}g_{12})+\mathcal{B}\mathcal{L}^{-1}(g_{21}-g_{22}+g_{31}-g_{32}). \tag{5.7}$$

We begin by calculating the inverse Laplace transform of function g_{11}. For this we can write g_{11} as

$$g_{11}=\frac{Ap+B}{\bar{\nu}p^2+\bar{\alpha}p+\bar{\beta}}+\frac{Cp+D}{\bar{\nu}p^2+(\bar{\alpha}+i\sqrt{\bar{\gamma}})p+\bar{\beta}}+\frac{Ep+F}{\bar{\nu}p^2+(\bar{\alpha}-i\sqrt{\bar{\gamma}})p+\bar{\beta}} \tag{5.8}$$

where A, B, C, D, E and F have the value

$$A=-\frac{\bar{\nu}\bar{\alpha}}{\bar{\beta}\bar{\gamma}}+\frac{\bar{\alpha}\bar{\nu}}{\bar{\beta}^2\bar{\gamma}},\quad B=-\frac{\bar{\alpha}}{\bar{\beta}\bar{\gamma}}\alpha+\frac{\bar{\alpha}^2}{\bar{\beta}^2\bar{\gamma}}-\frac{\bar{\nu}}{\bar{\beta}\bar{\gamma}},\quad C=\frac{\bar{\nu}\bar{\alpha}}{2\bar{\beta}\bar{\gamma}}+\frac{\bar{\nu}}{2\bar{\beta}^2 i\sqrt{\bar{\gamma}}}-\frac{\bar{\alpha}\bar{\nu}}{2\bar{\beta}^2\bar{\gamma}},$$

$$D=\frac{1}{2\bar{\beta}^2}+\frac{\bar{\alpha}}{2\bar{\beta}\bar{\gamma}}-\frac{\bar{\alpha}^2}{2\bar{\beta}^2\bar{\gamma}}+\frac{\bar{\nu}}{2\bar{\beta}\bar{\gamma}}-i\left(\frac{-\bar{\alpha}}{2\bar{\beta}\sqrt{\bar{\gamma}}}+\frac{\bar{\alpha}}{\bar{\beta}^2\sqrt{\bar{\gamma}}}\right),\quad E=\bar{C},\ F=\bar{D}. \tag{5.9}$$

Here we have denoted by $\bar{C}$ and $\bar{D}$ the complex-conjugate of C and D, respectively.

Using the tabel of inverse Laplace transform [7] we find

$$
\begin{aligned}
\mathcal{L}^{-1}g_{11} = {} & e^{-\frac{\bar{\alpha}}{2\bar{\nu}}t}\left[\frac{A}{\bar{\nu}}\cos at + \frac{1}{a\bar{\nu}}\left(B - \frac{A}{2\bar{\nu}}\right)\sin at\right] + \\
(5.10) \qquad & + e^{-\frac{\bar{\alpha}+i\sqrt{\bar{\gamma}}}{2\bar{\nu}}t}\left[\frac{C}{\bar{\nu}}\cos bt + \frac{1}{b\bar{\nu}}\left(D - C\,\frac{\bar{\alpha}+i\sqrt{\bar{\gamma}}}{2\bar{\nu}}\right)\sin bt\right] + \\
& + e^{-\frac{\bar{\alpha}-i\sqrt{\bar{\gamma}}}{2\bar{\nu}}}\left[\frac{E}{\bar{\nu}}\cos ct + \frac{1}{c\bar{\nu}}\left(E - F\,\frac{\bar{\alpha}-i\sqrt{\bar{\gamma}}}{2\bar{\nu}}\right)\sin ct\right],
\end{aligned}
$$

where

$$
(5.11) \qquad a^2 = \frac{4\bar{\nu}\bar{\beta} - \bar{\alpha}^2}{4\bar{\nu}^2}, \quad b^2 = \frac{4\bar{\nu}\bar{\beta} - \bar{\alpha}^2 - 2i\bar{\alpha}\sqrt{\bar{\gamma}} + \bar{\gamma}}{4\bar{\nu}^2}, \quad c^2 = \frac{4\bar{\nu}\bar{\beta} - \bar{\alpha}^2 + 2i\bar{\alpha}\sqrt{\bar{\gamma}} + \bar{\gamma}}{4\bar{\nu}^2}.
$$

Now we shall calculate the inverse Laplace transform of g_{12}. For this we denote by x_1, x_2 the roots of polynomial $\bar{\nu}p^2 + \bar{\alpha}p + \bar{\beta}$, by x_3, x_4 the roots of polynomial $\bar{\nu}p^2 + (\bar{\alpha} + i\sqrt{\bar{\gamma}})p + \bar{\beta}$ and by x_5, x_6 the roots of polynomial $\bar{\nu}p^2 + (\bar{\alpha} - i\sqrt{\bar{\gamma}})p + \bar{\beta}$, i.e.

$$
\begin{aligned}
& x_1 = -\frac{\bar{\alpha}}{2\bar{\nu}} - ia, \quad x_2 = -\frac{\bar{\alpha}}{2\bar{\nu}} + ia, \quad x_3 = -\frac{\bar{\alpha} + i\sqrt{\bar{\gamma}}}{2\bar{\nu}} - ib, \\
(5.12) \qquad & x_4 = -\frac{\bar{\alpha} + i\sqrt{\bar{\gamma}}}{2\bar{\nu}} + ib, \quad x_5 = -\frac{\bar{\alpha} - i\sqrt{\bar{\gamma}}}{2\bar{\nu}} - ic, \quad x_6 = -\frac{\bar{\alpha} - i\sqrt{\bar{\gamma}}}{2\bar{\nu}} + ic.
\end{aligned}
$$

Let λ_k be the roots of equation $J_0(\lambda) = 0$. For each $\lambda_k (k = 1, 2, \ldots)$ they correspond the roots of equation $J_0(\sqrt{\bar{M}_1} d) = 0$ given by

$$
(5.13) \qquad p_k = -\frac{\bar{\alpha}}{2\bar{\nu}} + i\sqrt{a^2 + \frac{\lambda_k^2}{d^2\bar{\nu}}}, \quad \bar{p}_k = -\frac{\bar{\alpha}}{2\bar{\nu}} - i\sqrt{a^2 + \frac{\lambda_k^2}{d^2\bar{\nu}}}.
$$

The function g_{12} contains Bessel function and therefore they are expressed as infinite series. We can write

$$
(5.14) \qquad g_{11}\frac{J_0(\sqrt{\bar{M}_1} r)}{J_0(\sqrt{\bar{M}_1} d)} = \sum_{k=1}^{6}\frac{A_k}{p - x_k} + \sum_{k=1}^{\infty}\left(\frac{B_k}{p - p_k} + \frac{\bar{B}_k}{p - \bar{p}_k}\right).
$$

In order to calculate the value of coefficients of series (5.14) we must remember the expression of Bessel function $J_0(x)$ [8] :

$$
(5.15) \qquad J_0(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!n!}\left(\frac{x}{2}\right)^{2n}.
$$

After some calculations we obtain

$$A_1=\frac{Ax_1+B}{\bar{\nu}(x_1-x_2)},\quad A_2=\frac{Ax_2+B}{\bar{\nu}(x_2-x_1)},\quad A_3=\frac{Cx_3+D}{\bar{\nu}(x_3-x_4)}\,\frac{J_0(\sqrt{M_1(x_3)}r)}{J_0(\sqrt{M_1(x_3)}d)},$$

$$A_4=\frac{Cx_4+D}{\bar{\nu}(x_4-x_3)}\,\frac{J_0(\sqrt{M_1(x_4)}r)}{J_0(\sqrt{M_1(x_4)}d)},\quad A_5=\frac{Ex_5+F}{\bar{\nu}(x_5-x_6)}\,\frac{J_0(\sqrt{M_1(x_5)}r)}{J_0(\sqrt{M_1(x_5)}d)},$$

(5.16)

$$A_6=\frac{Ex_6+F}{\bar{\nu}(x_6-x_5)}\,\frac{J_0(\sqrt{M_1(x_6)}r)}{J_0(\sqrt{M_1(x_6)}d)},$$

$$B_k=g_{11}(p_k)\,\frac{J_0(\sqrt{M_1(p_k)}r)}{J_0'(\sqrt{M_1(p_k)}d)},\quad \bar{B}_k=g_{11}(\bar{p}_k)\,\frac{J_0(\sqrt{M_1(\bar{p}_k)}r)}{J_0'(\sqrt{M_1(\bar{p}_k)}d)}.$$

The inverse Laplace transform of function g_{12} is

$$\mathcal{L}^{-1}g_{12}=e^{-\frac{\bar{\alpha}}{2\bar{\nu}}t}\Big\{(A_1+A_2)\cos at+i(A_2-A_1)\sin at+e^{-\frac{i\sqrt{\bar{\gamma}}t}{2\bar{\nu}}}[(A_3+$$

(5.17)

$$+A_4)\cos bt+i(A_4-A_3)\sin bt]+e^{\frac{i\sqrt{\bar{\gamma}}t}{2\bar{\nu}}}[(A_5+A_6)\cos ct+$$

$$+i(A_6-A_5)\sin ct]\Big\}+\sum_{k=0}^{\infty}(B_ke^{p_kt}+\bar{B}_ke^{\bar{p}_kt}).$$

Similarly, we decompose the function g_{21} in the sum

(5.18)
$$g_{21}=\frac{H_1}{p^3}+\frac{H_2}{p^2}+\frac{H_3}{p}+\frac{H_4p+H_5}{\bar{\nu}p^2+(\bar{\alpha}+i\sqrt{\bar{\gamma}})p+\bar{\beta}},$$

with $H_k(k=1, 2,..., 5$ given by

$$H_1=1,\quad H_2=\frac{1}{\varkappa\bar{\beta}}\left[1+\frac{\varepsilon}{2(1-\nu)}\right]-\frac{\bar{\alpha}}{\bar{\beta}},$$

(5.19)
$$H_3=\frac{1}{\varkappa\bar{\beta}}\left[\alpha^*+\frac{\varepsilon\alpha}{2(1-\nu)}\right]-\frac{\bar{\nu}}{\bar{\beta}}-\frac{H_2}{\bar{\beta}}(\bar{\alpha}+i\sqrt{\bar{\gamma}}),$$

$$H_4=-\bar{\nu}H_3,\quad H_5=-\bar{\nu}H_2-H_3(\bar{\alpha}+i\sqrt{\bar{\gamma}}).$$

The inverse Laplace transform of g_{21} is

$$\mathcal{L}^{-1}g_{21}=\frac{H_1}{2}t^2+H_2t+H_3+$$

(5.20)

$$+e^{-\frac{\bar{\alpha}+i\sqrt{\bar{\gamma}}}{2\bar{\nu}}t}\left[\frac{H_4}{\bar{\nu}}\cos bt+\frac{1}{b\bar{\nu}}\left(H_5-H_4\,\frac{\bar{\alpha}+i\sqrt{\bar{\gamma}}}{2\bar{\nu}}\right)\sin bt\right].$$

In order to calculate the inverse Laplace transform of function g_{22}, we must have the roots of equation $J_0(\sqrt{M_2}d)=0$. The roots of this equation are

(5.21)
$$\tilde{p}_k=\frac{\lambda_k^2}{i\sqrt{\bar{\gamma}}d^2}.$$

Writing

$$g_{21}\frac{J_0(\sqrt{i\sqrt{\bar{\gamma}}pr})}{J_0(\sqrt{i\sqrt{\bar{\gamma}}pd})}=\frac{C_1}{p^3}+\frac{C_2}{p^2}+\frac{C_3}{p}+\frac{C_4}{p-x_3}+\frac{C_5}{p-x_4}+\sum_{k=}^{\infty}\frac{D_k}{p-p_k}, \tag{5.22}$$

we calculate the coefficients C_1, C_2, C_3, C_4, C_5 and D_k $(k=1, 2,...)$ and they have the following values:

$$\begin{aligned}
&C_1=H_1,\quad C_2=H_2+H_1\frac{i\sqrt{\bar{\gamma}}}{2^2}(d^2-r^2),\\
&C_3=H_3-H_2\frac{i\sqrt{\bar{\gamma}}}{2^2}(r^2-d^2)-H_1\frac{\bar{\gamma}}{2^6}(r^4-d^4)-H_1\frac{\bar{\gamma}}{2^4}d^2(d^2-r^2),\\
&C_4=\frac{H_4x_3+H_5}{\bar{\nu}(x_3-x_4)}\,\frac{J_0(\sqrt{M_2(x_3)}r)}{J_0(\sqrt{M_2(x_3)}d)},\\
&C_5=\frac{H_4x_4+H_5}{\bar{\nu}(x_4-x_5)},\quad D_k=g_{21}(\tilde{p}_k)\frac{J_0(\lambda_k/d)}{J_0'(\lambda_k)}.
\end{aligned} \tag{5.23}$$

The inverse Laplace transform of function g_{22} is

$$\begin{aligned}
\mathcal{L}^{-1}g_{22}=&\frac{C_1}{2}t^2+C_2t+C_3+\\
&+e^{-\frac{\bar{\alpha}+i\sqrt{\bar{\gamma}}}{2\bar{\nu}}}[(C_4+C_5)\cos bt+i(C_5-C_4)\sin bt]+\sum_k D_k e^{p_k t}.
\end{aligned} \tag{5.24}$$

In the same way we shall calculate the inverse Laplace transforms of the functions g_{31} and g_{32} and find the expressions:

$$\begin{aligned}
\mathcal{L}^{-1}g_{31}=&\frac{\bar{H}_1}{2}t^2+\bar{H}_2t+\bar{H}_3+\\
&+e^{-\frac{\bar{\alpha}-i\sqrt{\bar{\gamma}}}{2}t}\left[\frac{\bar{H}_4}{\bar{\nu}}\cos ct+\frac{1}{c\bar{\nu}}\left(\bar{H}_5-\bar{H}_4\frac{\bar{\alpha}-i\sqrt{\bar{\gamma}}}{2\bar{\nu}}\right)\sin ct\right]
\end{aligned} \tag{5.25}$$

$$\begin{aligned}
\mathcal{L}^{-1}g_{32}=&\frac{\bar{C}_1}{2}t^2+\bar{C}_2+\bar{C}_3+\\
&+e^{-\frac{\bar{\alpha}-i\sqrt{\bar{\gamma}}}{2\nu}t}[(\bar{C}_4+\bar{C}_5)\cos ct+i(\bar{C}_5-\bar{C}_4)\sin ct[+\sum_k D_k e^{\bar{\tilde{p}}_k t},
\end{aligned} \tag{5.26}$$

where $\bar{H}_j$, $\bar{C}_j$ and $\bar{D}_k$ are the complex-conjugate of H_j and D_k, respectively. We have denoted by $\bar{\tilde{p}}_k$ the complex-conjugate of $\tilde{p}_k$.

Substituting (5.10), (5.17), (5.20), (5.24), (5.25) and (5.26) into (5.7) we find the expression of $w(r, t)$ given by

$$\begin{aligned}
w(r,t)=\mathcal{B}\Big[&\frac{H_1+\bar{H}_1-C_1-\bar{C}_1}{2}t^2+(H_2+\bar{H}_2-C_2-\bar{C}_2)t+(H_3+\bar{H}_3-\\
&-C_3-\bar{C}_3)\Big]+\mathcal{A}e^{-\frac{\bar{\alpha}}{2\bar{\nu}}t}\left\{\left(\frac{A}{\bar{\nu}}-A_1-A_2\right)\cos at+\left(\frac{B}{a\bar{\nu}}-\frac{A}{a\bar{\nu}^2}-iA_2+iA_1\right)\sin at+\right.
\end{aligned}$$

$$+e^{-\frac{i\sqrt{\bar{\gamma}}}{2\bar{\nu}}t}\left[\left(\frac{C}{\bar{\nu}}-A_3-A_4\right)\cos bt+\left(\frac{B}{b\bar{\nu}}+\frac{C}{b}\,\frac{\bar{\alpha}+i\sqrt{\bar{\gamma}}}{2\bar{\nu}^2}-iA_4+iA_3\right)\sin bt\right]+$$

$$+e^{\frac{i\sqrt{\bar{\gamma}}}{2\bar{\nu}}t}\left[\left(\frac{E}{\bar{\nu}}-A_5-A_6\right)\cos ct+\left(\frac{E}{c\bar{\nu}}-\frac{F}{c}\,\frac{\bar{\alpha}-i\sqrt{\bar{\gamma}}}{2\bar{\nu}^2}-iA_6+iA_5\right)\sin bt\right]\Bigg\}-$$

$$(2.27)\quad -\mathcal{A}\sum_{k=1}^{\infty}(B_k e^{p_k t}+\bar{B}_k e^{\bar{p}_k t})+\mathcal{B}e^{-\frac{\bar{\alpha}}{2\bar{\nu}}t}\Bigg\{e^{-\frac{i\sqrt{\bar{\gamma}}}{2\bar{\nu}}t}\left[\left(\frac{H_4}{\bar{\nu}}-C_4-C_5\right)\cos bt+\right.$$

$$\left.+\left(\frac{H_5}{b\bar{\nu}}-\frac{H_4}{b}\,\frac{\bar{\alpha}+i\sqrt{\bar{\gamma}}}{2\bar{\nu}^2}-iC_5+iC_6\right)\sin bt\right]+e^{\frac{i\sqrt{\bar{\gamma}}}{2\bar{\nu}}t}\left[\left(\frac{\bar{H}_4}{\bar{\nu}}-\bar{C}_4-\bar{C}_5\right)\cos ct+\right.$$

$$\left.\left.+\left(\frac{\bar{H}_5}{c\bar{\nu}}-\frac{\bar{H}_4}{c}\,\frac{\bar{\alpha}-i\sqrt{\bar{\gamma}}}{2\bar{\nu}^2}-\bar{C}_5+i\bar{C}_4\right)\sin ct\right]\Bigg\}-\mathcal{B}\sum_{k=1}^{\infty}(D_k e^{\tilde{p}_k t}+\bar{D}_k e^{\bar{\tilde{p}}_k t}).$$

6. We denote by $F(r, t)$ the function

$$(6.1)\qquad F(r,t)=\mathcal{B}\left[(H_1+\bar{H}_1-C_1-\bar{C}_1)\frac{t^2}{2}+(H_2+\bar{H}_2-C_2-\bar{C}_2)t+\right.$$
$$\left.+H_3+\bar{H}_3-C_3-\bar{C}_3\right].$$

Using (5.19) and (5.23) we find

$$(6.2)\qquad \begin{aligned} &H_1+H_1-C_1-C_1=0,\quad H_2+H_2-C_2-C_2=0,\\ &H_3+H_3-C_3-C_3=2\bar{\gamma}\left[\frac{1}{2^6}(d^4-r^4)-\frac{1}{2^4}d^2(d^2-r^2)\right].\end{aligned}$$

Substituting these values back in (6.1) we find that $F(r, t)$ is given by

$$(6.3)\qquad F(r,t)=-\frac{P}{N}\left[\frac{d^4-r^4}{2^6}-\frac{d^2(d^2-r^2)}{2^4}\right].$$

This expression is identified with the first term of the deflection $w(r, t)$ in the classical coupled theory of thermoelastic plates (E. I n a n [6]).

Received September 29, 1984

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REFERENCES

1. M ü l l e r I., Arch. Rational Mech. Anal., **41**, 319 (1971).
2. G r e e n A.E., L i n d s a y K.A., Journ. Elast., *1* (1972).
3. I b ă n e s c u I., Bul. Inst. polit. Iaşi, **XXVIII(XXXII)**, *1–4*, s. I (1982).
4. I b ă n e s c u I., Bul. Inst. polit. Iaşi, **XXIX(XXXIII)**, *1–4*, s. I (1983).
5. T i m o s h e n k o S.P., W o i n o v s k y-K r i e g e r S., *Theory of Plates and Shells*. Mc. Graw-Hill Book Comp,. Inc., New York—Toronto—London, 1959.
6. I n a n E., Acta Mech., **14** (1972).
7. S n e d d o n I., *Fourier Transforms*. New York—Toronto—London, 1951.
8. Ş a b a c I., *Matematici speciale*. Ed. did. şi ped., Buc., 1981.

VIBRAŢIILE PLĂCILOR CIRCULARE

(Rezumat)

Se studiază deformarea unei plăci circulare cu marginea încastrată elastic. Studiul deformării plăcii se face din punctul de vedere al teoriei termoelastice generalizate în sensul lui I. M ü l l e r şi A.E. G r e e n şi K.A. L i n d s a y.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 539.3 ; 532.13

INÉGALITÉS DISSIPATIVES ET VISCOÉLASTICITÉ LINÉAIRE MICROPOLAIRE

PAR

D. MANOLE

1. Introduction. Dans cet article on élargit dans la théorie linéaire de la viscoélasticité micropolaire une partie des résultats obtenus par Gurtin et Herrera [1] de la viscoélasticité linéaire. Par suite de la corrélation entre les fonctions caractéristiques et les fonctions de type positif [(3], le Théorème de Bochner, p. 159), toute une série de résultats obtenus de la théorie linéaire des probabilités peuvent être employés dans la théorie linéaire de la viscoélasticité micropolaire.

En voici les équations constitutives [2]:

$$
\begin{aligned}
t_{ij}(t) &= \int_0^t A_{ijkl}(t-\tau)\gamma_{kl}^{(1)}(\tau)\mathrm{d}\tau + \int_0^t B_{ijkl}(t-\tau)\varkappa_{kl}^{(1)}(\tau)\mathrm{d}\tau, \\
m_{ij}(t) &= \int_0^t B_{klij}(t-\tau)\gamma_{kl}^{(1)}(\tau)\mathrm{d}\tau + \int_0^t C_{ijkl}(t-\tau)\varkappa_{kl}^{(1)}(\tau)\mathrm{d}\tau,
\end{aligned}
\tag{1.1}
$$

où $\gamma_{ij}^{(1)}$, $\varkappa_{ij}^{(1)}$ sont les dérivés en rapport au temps des composantes cinématiques de la déformation, t_{ij}—les composantes du tenseur tension, m_{ij}—composantes du tenseur couple de tension, A_{ijkl}, B_{ijkl}, C_{ijkl} — fonctions de relaxation. Afin de simplifier l'exposé on va mettre, le plus souvent, les hypothèses de régularité imposées pour A_{ijkl}, γ_{ij},... . On va dire donc que les relations constitutives satisfont à la propriété de dissipativité si

$$
\int_0^t [t_{ij}(\tau)\gamma_{ij}^{(1)}(\tau) + m_{ij}(\tau)\varkappa_{ij}^{(1)}(\tau)]\mathrm{d}\tau \geqslant 0 \tag{1.2}
$$

pour $\gamma_{ij}(0)=0$, $\varkappa_{ij}(0)=0$.

Avec les notations de [1], on considère la loi intégrale $\mathcal{L}_\alpha$ définie par

$$
\mathcal{L}_\alpha: \quad
\begin{aligned}
\mathrm{t}(\alpha, t) &= \int_0^t A(t-s)\gamma_\alpha^{(1)}(s)\mathrm{d}s + \int_0^t B(t-s)\varkappa_\alpha^{(1)}(s)\mathrm{d}s, \\
\mathrm{m}(\alpha, t) &= \int_0^t B(t-s)\gamma_\alpha^{(1)}(s)\mathrm{d}s + \int_0^t C(t-s)\varkappa_\alpha^{(1)}(s)\mathrm{d}s, \quad (0 \leqslant t < \infty),
\end{aligned}
\tag{1.3}
$$

où $\gamma_\alpha, \varkappa_\alpha$ représentent l'histoire cinématique de la déformation micropolaire, $\mathbf{t}$ – l'histoire de la tension, $\mathbf{m}$ — l'histoire du couple de tension, A, B, C — des fonctions de relaxation.

Soit l'esspace $\mathcal{A}_\alpha$ défini par

$\mathcal{A}_\alpha = \{\gamma_\alpha, \varkappa_\alpha \mid \gamma_\alpha, \varkappa_\alpha$ sont continuels sur $[0, \infty)$ avec $\gamma_\alpha(0)=0, \varkappa_\alpha(0)=0\}$.

Pour chaque $\gamma_\alpha, \varkappa_\alpha \in \mathcal{A}_\alpha$ on définit

$$w_{\gamma_\alpha \varkappa_\alpha}(t) = \int_0^t [\mathbf{t}(\alpha, s)\gamma_\alpha^{(1)}(s) + \mathbf{m}(\alpha, s)\varkappa_\alpha^{(1)}(s)]\mathrm{d}s, \quad (0 \leqslant t < \infty) \tag{1.4}$$

où

$$\gamma_\alpha(t) = \gamma(\alpha t) \quad \varkappa_\alpha(t) = \varkappa(\alpha t), \quad (0 \leqslant t < \infty). \tag{1.5}$$

Conventionnellement, si α n'apparaît pas dans les γ_α, $\varkappa_\alpha$ alors on va noter

$$\gamma(t),\ \varkappa(t),\ \mathbf{t}(t), \mathbf{m}(t), w_{\gamma\varkappa}(t), \mathcal{A}, \mathcal{L}.$$

On va dire que $\mathcal{L}$ est dissipative si et seulement si

$$w_{\gamma\varkappa}(t) \geqslant 0, \ (0 \leqslant t < \infty) \text{ pour chaque } \gamma, \varkappa \in \mathcal{A}. \tag{1.6}$$

2. La dissipativité des relations constitutives. Théorème 2.1. *Si* $\gamma, \varkappa \in \mathcal{A}$ *alors*

$$\lim_{\alpha \to \infty} w_{\gamma_\alpha \varkappa_\alpha}\left(\frac{t}{\alpha}\right) = \int_0^t \gamma^{(1)}(s)[A(0)\gamma(s) + B(0)\varkappa(s)]\mathrm{d}s + \int_0^t \varkappa^{(1)}(s)[B(0)\gamma(s) + C(0)\varkappa(s)]\mathrm{d}s. \tag{2.1}$$

Démonstration. De (1.4) on obtient

$$w_{\gamma_\alpha \varkappa_\alpha}\left(\frac{t}{\alpha}\right) = \int_0^{\frac{t}{\alpha}} [\mathbf{t}(\alpha, s)\gamma_\alpha^{(1)}(s) + \mathbf{m}(\alpha, s)\varkappa_\alpha^{(1)}(s)]\mathrm{d}s, \quad (0 \leqslant t < \infty). \tag{2.2}$$

Si on fait en (2.2) un changement de variable et l'on tient compte de (1.5) on obtient

$$w_{\gamma_\alpha \varkappa_\alpha}\left(\frac{t}{\alpha}\right) = \int_0^t \left[\mathbf{t}\left(\alpha, \frac{s}{\alpha}\right)\gamma^{(1)}(s) + \mathbf{m}\left(\alpha, \frac{s}{\alpha}\right)\varkappa^{(1)}(s)\right]\mathrm{d}s, \tag{2.3}$$

où

$$\mathbf{t}\left(\alpha, \frac{s}{\alpha}\right) = \int_0^s \left[A\left(\frac{s-\mu}{\alpha}\right)\gamma^{(1)}(\mu) + B\left(\frac{s-\mu}{\alpha}\right)\varkappa^{(1)}(\mu)\right]\mathrm{d}\mu, \tag{2.4}$$

$$\mathbf{m}\left(\alpha, \frac{s}{\alpha}\right) = \int_0^s \left[B\left(\frac{s-\mu}{\alpha}\right)\gamma^{(1)}(\mu) + C\left(\frac{s-\mu}{\alpha}\right)\varkappa^{(1)}(\mu)\right]\mathrm{d}\mu, \quad (0 \leqslant s < \infty).$$

Puisque $\gamma, \varkappa \in \mathcal{A}$ de (2.4) résulte

$$\lim_{\alpha\to\infty} \mathbf{t}\left(\alpha, \frac{s}{\alpha}\right) = \int_0^s [A(0)\gamma^{(1)}(\mu) + B(0)\varkappa^{(1)}(\mu)]d\mu = A(0)\gamma(s) + B(0)\varkappa(s),$$

(2.5)

$$\lim_{\alpha\to\infty} \mathbf{m}\left(\alpha, \frac{s}{\alpha}\right) = \int_0^s [B(0)\gamma^{(1)}(\mu) + C(0)\varkappa^{(1)}(\mu)]d\mu = B(0)\gamma(s) + C(0)\varkappa(s),$$

et par conséquent (2.3) et (2.5) impliquent (2.1).

Théorème 2.2. *Si*

$$A(\infty) = \lim_{t\to\infty} A(t), \quad B(\infty) = \lim_{t\to\infty} B(t), \quad C(\infty) = \lim_{t\to\infty} C(t)$$

existent, alors pour $\gamma, \varkappa \in \mathcal{A}$

$$\lim_{\alpha\to 0} w_{\gamma_\alpha \varkappa_\alpha}\left(\frac{t}{\alpha}\right) = \int_0^t \gamma^{(1)}(s)[A(\infty)\gamma(s) + B(\infty)\varkappa(s)]ds + \tag{2.6}$$

$$+ \int_0^t \varkappa^{(1)}(s)[B(\infty)\gamma(s) + C(\infty)\varkappa(s)]\, ds.$$

Démonstration. Puisque $\gamma, \varkappa \in \mathcal{A}$ de (2.4) on obtient

$$\lim_{\alpha\to 0} \mathbf{t}\left(\alpha, \frac{s}{\alpha}\right) = \int_0^s [A(\infty)\gamma^{(1)}(\mu) + B(\infty)\varkappa^{(1)}(\mu)]d\mu = A(\infty)\gamma(s) + B(\infty)\varkappa(s),$$

(2.7)

$$\lim_{\alpha\to 0} \mathbf{m}\left(\alpha, \frac{s}{\alpha}\right) = \int_0^s [\beta(\infty)\gamma^{(1)}(\mu) + C(\infty)\varkappa^{(1)}(\mu)]d\mu = B(\infty)\gamma(s) + C(\infty)\varkappa(s)$$

et par conséquent (2.3) et (2.7) impliquent (2.6).

Théorème 2.3. *Si* $\mathcal{L}$ *est dissipative, alors* $A(0)$, $C(0)$ *sont symétriques et*

$$\eta A(0)\eta + 2\eta B(0)\eta + \eta C(0)\eta \geqslant 0, \tag{2.8}$$

où

$$\eta(s) = [\gamma(s)\varkappa(s)].$$

Démonstration. Soit $F(0) = \begin{bmatrix} A(0) & B(0) \\ B(0) & C(0) \end{bmatrix}$. Étant données les notations faites, de (1.7) et (2.1) on obtient

$$\int_0^t \eta^{(1)}(s)F(0)\eta(s)ds \geqslant 0, \quad (0 \leqslant t < \infty). \tag{2.9}$$

En appliquant le Théorème 3.1 [1] on obtient le fait que $F(0)$ est simétrique et positif sémi-définie. Mais, conformément à la notation faite, $F(0)$ est symétrique si et seulement si $A(0)$ et $C(0)$ sont également symétriques. Le fait que $F(0)$ est positive sémi-définien implique (2.8) et par conséquent le Théorème est démontré.

Théorème 2.4. *Si $\mathcal{L}$ est dissipative et si $\mathcal{A}(\infty)$, $B(\infty)$, $C(\infty)$ existent, dans ce cas $A(\infty)$, $B(\infty)$, $C(\infty)$ sont également symétriques et*

$$\eta A(\infty)\eta+2\eta B(\infty)\eta+\eta C(\infty)\eta\geqslant 0. \tag{2.10}$$

Démonstration. Dans le Théorème 2.3 au lieu de $A(0)$, $B(0)$, $C(0)$ on va prendre $A(\infty)$, $B(\infty)$, $C(\infty)$ en cas d'existence. Soit

$$g_\theta(t)=\theta A(|t|)\theta+2\theta B(|t|)\theta+\theta C(|t|)\theta, \qquad \theta\in R^N, \quad -\infty<t<\infty.$$

Théorème 2.5. *Si $\mathcal{L}$ est dissipative alors g_θ est de type positif.*

Démonstration. Soit $\gamma(t)=\theta\int_0^t f(s)\mathrm{d}s$, $\varkappa(t)=\theta\int_0^t f(s)\mathrm{d}s$ où f est une fonction continuelle définie par $[0,\infty)$. Il est évident que $\gamma,\varkappa\in\mathcal{A}$. En remplaçant (1.3) dans (1.4) il résulte

$$\begin{aligned} w_{\gamma\varkappa}(t)=\tfrac{1}{2}\int_0^t\int_0^s \gamma^{(1)}(s)A(s-\tau)\gamma^{(1)}(\tau)\mathrm{d}s\mathrm{d}\tau+\int_0^t\int_0^s \gamma^{(1)}(s)B(s-\tau)\varkappa^{(1)}(\tau)\mathrm{d}s\mathrm{d}\tau+ \\ +\int_0^t\int_0^s \varkappa^{(1)}(s)B(s-\tau)\gamma^{(1)}(\tau)\mathrm{d}s\mathrm{d}\tau+\int_0^t\int_0^s \varkappa^{(1)}(s)C(s-\tau)\varkappa^{(1)}(\tau)\mathrm{d}s\mathrm{d}\tau. \end{aligned} \tag{2.11}$$

Compte tenu du choix pour $\gamma(s)$ et $\varkappa(s)$ et ainsi que $\mathcal{L}$ est dissipative, de (2.11) on obtient

$$w_{\gamma\varkappa}(t)=\tfrac{1}{2}\int_0^t\int_0^t f(s)f(\tau)g_\theta(s-\tau)\mathrm{d}s\mathrm{d}\tau, \quad (0\leqslant t<\infty), \tag{2.12}$$

d'où on déduit que

$$\int_{-t}^{t}\int_{-t}^{t} f(s+t)f(\tau+t)g_\theta(s-\tau)\mathrm{d}s\mathrm{d}\tau\geqslant 0, \qquad (0\leqslant t<\infty)$$

et conformément au Lemme 4.4 [1] il résulte que g_θ est de type positif.

Théorème 2.6. *Si $\mathcal{L}$ est dissipative alors pour $(0\leqslant t<\infty)$ et $\theta\in\mathbb{R}^N$ il résulte que $\theta A(t)\theta+2\theta B(t)\theta+\theta C(t)\theta$ est une fonction continuelle de t et*

$$|\,\theta A(t)\theta+2\theta B(t)\theta+\theta C(t)\theta\,|\leqslant\theta A(0)\theta+2\theta B(0)\theta+\theta C(0)\theta.$$

Démonstration. Ce Théorème en résulte facilement si l'on tient compte du Théorème 2.5 et Lemme 4.2 [1].

Théorème 2.7. *Si A, B, C sont par deux fois dérivables sur $[0,\infty)$ avec $A(0)$ et $C(0)$ symétriques et*

$$\eta A(0)\eta+2\eta B(0)\eta+\eta C(0)\eta>0 \tag{2.13}$$

alors $\gamma(t)\equiv 0$, $\varkappa(t)\equiv 0$ pour $t\geqslant 0$.

D é m o n s t r a t i o n Si $\gamma, \varkappa \in \mathcal{A}$ les relations constructives (1.3) peuvent etre mises sous la forme

$$
\begin{aligned}
\mathbf{t}(t) &= A(0)\gamma(t) + B(0)\varkappa(t) + \int_0^t A^{(1)}(t-s)\gamma(s)\mathrm{d}s + \int_0^t B^{(1)}(t-s)\varkappa(s)\mathrm{d}s, \\
\mathbf{m}(t) &= B(0)\gamma(t) + C(0)\varkappa(t) + \int_0^t B^{(1)}(t-s)\gamma(s)\mathrm{d}s + \int_0^t C^{(1)}(t-s)\varkappa(s)\mathrm{d}s, \\
&(0 \leqslant t < \infty).
\end{aligned} \tag{2.14}
$$

En remplaçant (2.14) en (1.4) et en intégrant par parties, on obtient

$$
\begin{aligned}
&\tfrac{1}{2}\gamma(t)A(0)\gamma(t) + \gamma(t)B(0)\varkappa(t) + \tfrac{1}{2}\varkappa(t)C(0)\varkappa(t) + \\
&+ \int_0^t \gamma(t)A^{(1)}(t-\tau)\gamma(\tau)\mathrm{d}\tau + \int_0^t \varkappa(t)C^{(1)}(t-\tau)\varkappa(\tau)\mathrm{d}\tau + \int_0^t \gamma(t)B^{(1)}(t-\tau)\varkappa(\tau)\mathrm{d}\tau + \\
&+ \int_0^t \varkappa(t)B^{(1)}(t-\tau)\gamma(\tau)\mathrm{d}\tau + \int_0^t \gamma(\tau)A^{(1)}(0)\gamma(\tau)\mathrm{d}\tau - \int_0^t \varkappa(\tau)C^{(1)}(0)\varkappa(\tau)\mathrm{d}\tau - \\
&- \int_0^t \gamma(\tau)B^{(1)}(0)\varkappa(\tau)\mathrm{d}\tau - \int_0^t \varkappa(\tau)B^{(1)}(0)\gamma(\tau)\mathrm{d}\tau - \\
&- \int_0^t\int_0^\tau \gamma(\tau)A^{(2)}(t-s)\gamma(s)\mathrm{d}s\mathrm{d}\tau - \int_0^t\int_0^\tau \varkappa(\tau)C^{(2)}(t-s)\varkappa(s)\mathrm{d}s\mathrm{d}\tau - \\
&- \int_0^t\int_0^\tau \varkappa(\tau)B^{(2)}(t-s)\gamma(s)\mathrm{d}s\mathrm{d}\tau - \int_0^t\int_0^\tau \gamma(\tau)B^{(2)}(t-s)\varkappa(s)\mathrm{d}s\mathrm{d}\tau = 0.
\end{aligned} \tag{2.15}
$$

La relation (2.15) est équivalente à

$$
\begin{aligned}
&\eta^{\mathsf{T}}(t)\frac{F(0)}{2}\eta(t) + \int_0^t \eta^{\mathsf{T}}(t)F^{(1)}(t-\tau)\eta(\tau)\mathrm{d}\tau - \\
&- \int_0^t \eta^{\mathsf{T}}(\tau)F^{(1)}(0)\eta(\tau)\mathrm{d}\tau - \int_0^t\int_0^\tau \eta^{\mathsf{T}}(\tau)F^{(2)}(t-s)\eta(s)\mathrm{d}s\,\mathrm{d}\tau = 0,
\end{aligned} \tag{2.16}
$$

qui est analogue à la relation du Théorème 5.2 donné dans [1].

De (2.13) il résulte que $\delta > 0$ existe de sorte que

$$\eta^{\mathsf{T}}(t)F(0)\eta(t) \geqslant \delta \mid \eta(t) \mid^2.$$

Si on note [1]

$$\omega(t) = |\eta(t)|, \qquad \beta(t_0) = \frac{2}{\delta}\sup_{0 \leqslant t_0 \leqslant t}(\|F^{(1)}(t)\| + \|F^{(2)}(t)\|), \tag{2.17}$$

dans ce cas $\beta(t_0) < \infty$.

Compte tenu du fait que

$$[\tau^{r/2}\omega(t)-t^{r/2}\omega(\tau)]\geqslant 0 \tag{2.18}$$

et ainsi que de l'inégalité de Schwartz, il résulte par induction que

$$\omega^2(t)\leqslant \rho(t_0)\,\frac{[K(t_0)t]^m}{m!}\qquad (m=0,\,1,\,2,\ldots), \tag{2.19}$$

où

$$K(t_0)=\beta(t_0)\left(3+\frac{t_0}{2}\right),\quad \rho(t_0)=\sup_{0\leqslant t\leqslant t_0}\omega^2(t). \tag{2.20}$$

De (2.19) résulte que $\eta(t)\equiv 0$ pour $t\geqslant 0$ et donc le Théorème est démontré.

Reçue le 28 Mars 1980

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BIBLIOGRAPHIE

1. Gurtin M.E., Herrera I., *On Dissipation Inequalities and Linear Viscoelasticity*. Quart. Appl. Math., **23**, 235–245 (1965).
2. Ieşan D., *Sur la théorie de la viscoélasticité micropolaire*. C. Rend. Acad. Sc. Paris, **270**, 140–143 (1970).
3. Mihoc Gh., Ciucu G., Craiu V., *Teoria probabilităţilor şi statistică matematică*. Ed. did. şi ped., Buc., 1970.

INEGALITĂŢI DISIPATIVE ŞI VÎSCOELASTICITATEA LINIARĂ MICROPOLARĂ

(Rezumat)

Se extind în teoria liniară a vîscoelasticităţii micropolare o parte din rezultatele obţinute de Gurtin şi Herrera [1] în teoria vîscoelasticităţii liniare.

Datorită legăturii dintre funcţiile de tip pozitiv şi funcţiile caracteristice o serie de rezultate din teoria probabilităţilor pot fi utilizate în teoria liniară a vîscoelasticităţii micropolare.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 539.13 ; 531.36

LIAPUNOV STABILITY IN THE LINEAR GENERALIZED THEORY OF SHELLS

BY

GH. CHIORESCU

1. Introduction. The stability in the linear generalized theory of shells (called also Cosserat surfaces or C-surfaces) was developped in the papers [1]—[4]. The Liapunov stability is studied in the paper [3], where the case of isotropic plates is considered. In this paper we prove a stability theorem in the case of anisotropic shells.

2. Basic Equations [5]. *C-surface.* By C-surface we mean a kinematical model constitued by a material surface embedded in Euclidean 3-space, to every point of which it is assigned a vector called director. Such that a present configuration at time t of a C-surface is given by a couple of mappings

$$\mathbf{r}=\mathbf{r}(\theta^1, \theta^2, t), \quad \mathbf{d}=\mathbf{d}(\theta^1, \theta^2, t), \quad [\mathbf{a}_1, \mathbf{a}_2, \mathbf{d}]>0, \quad \mathbf{a}=\mathbf{r}_{,\alpha}$$

from an open bounded set $\Omega\subset\mathbb{R}^2$ into an Euclidean 3-space, where θ^α, $\alpha=1, 2$ are the surface parameters. The initial configuration ($t=0$) of a C-surface is given by the couple of mappings

$$\mathbf{R}=\mathbf{R}(\theta^1, \theta^2), \quad \mathbf{D}=\mathbf{D}(\theta^1, \theta^2).$$

Let $\mathbf{A}_\alpha$, $\mathbf{A}_3$ be given by

$$\mathbf{A}_3=\frac{\partial\mathbf{R}}{\partial\theta^\alpha}, \quad \mathbf{A}_3=\frac{\mathbf{A}_1\times\mathbf{A}_2}{|\mathbf{A}_1\times\mathbf{A}_2|}.$$

We suppose that $\mathbf{D}=D(\theta^1, \theta^2)\mathbf{A}_3$. In this mathematical model of shells the vector $\mathbf{d}$ represents the variation of the transversal section.

i) *Geometrical equations*

$$\mathbf{r}=\mathbf{R}+\mathbf{u}, \quad \mathbf{d}=\mathbf{D}+\delta, \tag{1}$$

$$\begin{aligned}
&e_{\alpha\gamma}=\tfrac{1}{2}\,(u_{\alpha/\gamma}+u_{\gamma/\alpha})-B_{\alpha\gamma}u_3,\\
&\varkappa_{\gamma\alpha}=D_{,\alpha}B^\nu_\gamma u_\nu+DB^\beta_\alpha B_{\gamma\beta}u_3-B_{\alpha\gamma}\delta_3-DB^\beta_\alpha u_{\beta/\gamma}+D_{,\alpha}u_{3,\gamma}+\delta_{\gamma/\alpha},\\
&s_\alpha=D^2B^\nu_\alpha B^\delta_\nu u_\delta+DB^\nu_\alpha\delta_\nu+D^2B^\nu_\alpha u_{3,\nu}+D\delta_{3,\alpha}+D_{,\alpha}\delta_3,\\
&\gamma_\alpha=DB^\nu_\alpha u_\nu+\delta_\alpha+Du_{3,\alpha},\\
&s\ =2D\delta_3,
\end{aligned} \tag{2}$$

where $\mathbf{u}$ and δ are the displacement vectors and $e_{\alpha\gamma}$, $\varkappa_{\gamma\alpha}$, s_α, γ_α, s are the deformation measures. A comma denotes the partial derivative and a vertical line stands for the covariant derivative with respect to the metric tensor $A_{\alpha\beta}$ on $\mathcal{S}$. $B_{\alpha\beta}$ are the components of the second fundamental form of $\mathcal{S}$. All vector or tensor valued functions are represented in the base $(\mathbf{A}_\alpha, \mathbf{A}_3)$ or its dual $(\mathbf{A}^\alpha, \mathbf{A}^3)$ defined on $\mathcal{S}$.

iii) *Constitutive equations*

$$
\begin{aligned}
&N^{\alpha\beta} + DMB^\beta_\gamma = \rho_0 \frac{\partial\psi}{\partial e_{\alpha\beta}}, \\
&m^\alpha = \rho_0 \frac{\partial\psi}{\partial\gamma_\alpha}, \quad m^3 = \rho_0\left(2D\frac{\partial\psi}{\partial s} + D_{,\alpha}\frac{\partial\psi}{\partial s_\alpha}\right), \\
&M^{\alpha\beta} = \rho_0 \frac{\partial\psi}{\partial\varkappa_{\beta\alpha}}, \quad M^{\alpha 3} = \rho_0 D\frac{\partial\psi}{\partial s_\alpha}. \\
&N^{\alpha 3} - m^\alpha D - M^{\gamma 3} D B^\alpha_\gamma - M^{\gamma\alpha} D_{,\gamma} = 0,
\end{aligned}
\tag{3}
$$

where ψ is the deformation energy function, $N^{\alpha i}$, $M^{\alpha i}$, m^i are the components of the tension tensors, ρ_0 is the reference configurations density.

iii) *The motion equations*

$$
\begin{aligned}
&N^{\alpha\beta} - B^\beta_\alpha N^{\alpha 3} + \rho_0 F^\beta = \rho_0 \ddot{u}^\beta, \\
&N^{\beta 3}_{/\alpha} + B_{\alpha\beta} N^{\alpha\beta} + \rho_0 F^3 = \rho_0 \ddot{u}^3, \\
&M^{\alpha\beta}_{/\alpha} - B^\beta_\alpha M^{\alpha 3} + \rho_0 L^\beta - m^\beta = \rho_0 \alpha \ddot{\delta}^\beta, \\
&M^{\alpha 3}_{/\alpha} + B_{\alpha\beta} M^{\alpha\beta} + \rho_0 L^3 - m^3 = \rho_0 \alpha \ddot{\delta}^3,
\end{aligned}
\tag{4}
$$

where F^i and L^i are the components of the body forces and α is the inertia coefficient. We suppose that the function $\rho_0\psi$ is a quadratic form in deformations

$$
\begin{aligned}
\rho_0\psi = {}&{}_1C^{\alpha\beta\gamma\delta} e_{\alpha\beta} e_{\gamma\delta} + {}_2C^{\alpha\beta\gamma\delta} \varkappa_{\alpha\beta} \varkappa_{\gamma\delta} + {}_3C^{\alpha\beta\gamma\delta} e_{\alpha\beta} \varkappa_{\delta\gamma} + {}_1C^{\alpha\beta\gamma} s_\alpha \varkappa_{\beta\gamma} + \\
&+ {}_2C^{\alpha\beta\gamma} e_\alpha \gamma_\gamma + {}_3C^{\alpha\beta\gamma} e_{\alpha\beta} s_\gamma + {}_4C^{\alpha\beta\gamma} \gamma_\alpha \varkappa_{\beta\gamma} + {}_1C^{\alpha\beta} \gamma_\alpha \gamma_\beta + {}_2C^{\alpha\beta} s_\alpha s_\beta + \\
&+ {}_3C^{\alpha\beta} \gamma_\alpha s_\beta + {}_4C^{\alpha\beta} e_{\alpha\beta} s + {}_5C^{\alpha\beta} \varkappa_{\alpha\beta} s + {}_1C^\alpha \gamma_\alpha s + {}_2C^\alpha s_\alpha s + C(s)^2,
\end{aligned}
\tag{5}
$$

where the coefficients satisfy the following conditions:

$$
\begin{aligned}
&{}_1C^{\alpha\beta\gamma\delta} = {}_1C^{\beta\alpha\gamma\delta} = {}_1C^{\gamma\delta\alpha\beta} = {}_1C^{\alpha\beta\gamma\delta}, \quad {}_2C^{\alpha\beta\gamma\delta} = {}_2C^{\gamma\delta\alpha\beta}, \quad {}_3C^{\alpha\beta\gamma\delta} = {}_3C^{\beta\alpha\gamma\delta}, \\
&{}_2C^{\alpha\beta\gamma} = {}_2C^{\beta\alpha\gamma}, \quad {}_3C^{\alpha\beta\gamma} = {}_3C^{\beta\alpha\gamma}, \quad {}_4C^{\alpha\beta\gamma} = {}_4C^{\beta\gamma\alpha}, \quad {}_1C^{\alpha\beta} = {}_1C^{\beta\alpha}, \\
&{}_2C^{\alpha\beta} = {}_2C^{\beta\alpha}, \quad {}_3C^{\alpha\beta} = {}_3C^{\beta\alpha}; \quad {}_4C^{\alpha\beta} = {}_4C^{\beta\alpha}.
\end{aligned}
\tag{6}
$$

3. The Dynamical System. The Movchan-Liapunov Functional. Metrics. Let $\mathbf{h} = (u_1, u_2, u_3, \delta_1, \delta_2, \delta_3)$ be the displacement vector and let X be the subset of functions from $C^2(\mathcal{S})$ which satisfy the condition

$$
\mathbf{h} = 0, \quad u_{1/2} - u_{2/1} = 0 \quad \text{on} \quad \partial\mathcal{S}. \tag{C$_1$}
$$

D e f i n i t i o n. We call dynamical system for the above boundary value problem, the set of functions defined on $T=[0, \infty)$ taking values in X and satisfying the Eqs. (1) ... (5). We denote by $\mathcal{F}(t, X)$ this set.

Without loss in generality we study the stability of the null solution, $\mathbf{u}=\delta=\mathbf{o}$.

We consider the following Movchan-Liapunov functional on $\mathcal{F}(T, X)$:

$$F(\mathbf{h}, \mathbf{o})=\sup_{t\in T} F(t), \quad F(t)=\int_{\mathcal{S}} \rho_0\left(\psi+\frac{\dot{\mathbf{u}}.\dot{\mathbf{u}}}{2}+\alpha\frac{\dot{\delta}.\dot{\delta}}{2}\right)\mathrm{d}\Sigma. \tag{7}$$

As a measure of the perturbations we use the metrics

$$\mu(\mathbf{h}, \mathbf{o})=\sup_{t\in T}\mu(t), \quad \mu(t)=\int_{\mathcal{S}}(\mathbf{u}.\mathbf{u}+\delta.\delta+\dot{\mathbf{u}}.\dot{\mathbf{u}}+\alpha\dot{\delta}.\dot{\delta})\mathrm{d}\Sigma, \quad \mu_0=F(0). \tag{8}$$

4. Liapunov Stability Theorem. L e m m a 1. *If the condition* (C_1) *and the condition*

(C_2) *the body forces are zero,*

hold, then $F(t)$ is constant.

P r o o f. By using the Eqs. (2) ... (4) and Stoke's formula we get

$$\dot{F}(t)=\int_{\partial\mathcal{S}}(N^{\alpha i}u_i+M^{\alpha i}\dot{\delta}_i)_0\nu_\alpha \mathrm{d}s+\int_{\mathcal{S}}\rho_0(\mathbf{F}.\dot{\mathbf{u}}+\mathbf{L}.\dot{\delta})\mathrm{d}\Sigma=0. \tag{9}$$

L e m m a 2. *If the following conditions are satisfied :*

(C_3) *the coefficients ${}_iC$, $i=\overline{0, 5}$ are measurable bounded functions ;*

(C_4) *the function ψ is positive defined i.e.*

$$\int_{\mathcal{S}}\rho_0\psi\mathrm{d}\Sigma\geqslant c\int_{\mathcal{S}}\left[\sum_{\alpha,\beta=1}^{2}(e_{\alpha\beta}^2+\varkappa_{\alpha\beta}^2)+\sum_{\alpha=1}^{2}(\gamma_\alpha^2+s_\alpha^2)+s^2\right]\mathrm{d}\Sigma, \quad c>0, c=\text{const.}; \tag{10}$$

(C_5) $D(\theta^1, \theta^2)\in C^1(\Omega)$ *and* $B_{\alpha\beta}(\theta^1, \theta^2)\in C^0(\Omega)$,

then in X the following inequality

$$\int_{\mathcal{S}}\rho_0\psi\mathrm{d}\Sigma\geqslant c\int_{\mathcal{S}}\sum_{i=1}^{6}h_i^2\mathrm{d}\Sigma, \quad c>0, \quad c=\text{const.} \tag{11}$$

holds.

P r o o f. In the paper [6] was proved the inequality (Lemma 4.2)

$$\int_{\mathcal{S}}\rho_0\psi\mathrm{d}\Sigma\geqslant\|h\|_1^2, \quad \text{where} \quad \|h\|_1^2=\int_{\mathcal{S}}\left(\sum_1^6 h_i^2\right)\mathrm{d}\Sigma+\int_{\mathcal{S}}\left(\sum_{\alpha,i=1}^{6,2}h_{i,\alpha}^2\right)\mathrm{d}\Sigma.$$

From this the inequality (11) is obvious.

T h e o r e m (*Stability Theorem*). *If the conditions* (C_i) $i=\overline{1, 5}$ *are satisfied then the null solution of the problem is stable with respect to the functional* (7) *and the metrics* (8).

Proof. From the inequality (11) it results that $F(t) > c\mu(t)$, $c > 0$, $c = \text{const}$. Since $\mu_0 = F(0) = F(t)$ the proof results in conformity to the stability theorem of Liapunov [7].

Received December 10, 1984

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REFERENCES

1. Popelar C.N., *Theory of Stability of an Elastic Cosserat Surface*, Int. J. Eng. Sci., **8**, 529 (1970).
2. Dikman M., *Stability of the Cosserat Surface*. I.U.T.A.M. Symp. on „Instability of Continuous System", Springer, Berlin–Heidelberg–New-York, 1971.
3. Leipholz H.H.E., *Stability of Elastic Plates via Liapunov's Second Method*. Ingenieur Arch., **45**, 337–345 (1976).
4. Shahinpoor M., *Plane Waves and Stability of Thin Circular Cylindrical Shells*. Arch. Rational Mech. Anal., **54**, 267–280 (1974).
5. Chiorescu Gh., *Boundary Value Problems in the Generalized Theory of Thin Bodies*. Thesis, University „Al.I. Cuza", Jassy, 1980 (in Romanian).
6. Chiorescu Gh., *Weak Solutions in Generalized Theory of Shell with Non-uniform Thickness*. Bul. Inst. polit. Iaşi, **XXIX(XXXIII)**, *1–4*, s. I (1983).
7. Knops R.J., Wilkes E.W., *Theory of Elastic Stability*. In *Handbuch der Physik*, Vol. **VI**$_{a/3}$, Springer Verlag, Berlin–Heidelberg–New-York, 1973.

STABILITATE LIAPUNOV ÎN TEORIA LINIARĂ GENERALIZATĂ A PLĂCILOR CURBE

(Rezumat)

Se consideră problema la limită în deplasări din teoria liniară generalizată a plăcilor curbe anizotrope cu grosime neuniformă. Presupunînd că deplasările sînt nule pe frontieră, forţele masice sînt absente, coeficienţii de elasticitate sînt funcţii măsurabile şi mărginite, funcţia energie de deformare este pozitiv definită, funcţia care reprezintă grosimea plăcii este continuă şi are derivate de ordinul întîi continue şi suprafaţa modelului generalizat de placă are curbura continuă—se demonstrează o teoremă de stabilitate Liapunov a soluţiei zero.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 532.5

DECAY OF THE KINETIC ENERGY OF INCOMPRESSIBLE MICROPOLAR FLUIDS

BY

VALERIU AL. SAVA

1. Introduction. The theory of micropolar fluids [1] introduced by Eringen in 1966 differs from the classical theory of Navier-Stokes viscous fluids in two important features, viz. the sustenance of the couple stress in the fluid medium and nonsymmetry of the force stress tensor. In this theory the fluid element has the usual translatory degrees of freedom reckoned by the velocity vector $\mathbf{q}$ and has, in addition, degrees of freedom, enabling the intrinsic rotary motions of the fluid element. The latter motion is reckogned by the microrotation vector ν. The constitutive equations for the stress and couple stress in the case of a non-heat conducting micropolar fluid medium have been presented by Eringen and when the model is assumed to be linear, these involve, in all, six material constants.

We must mention here that the analysis on the decay of kinetic energy of incompressible micropolar fluids was first reported by Lakshmana Rao [5].

We consider the model for micropolar fluid under certain simplifying assumptions :

a) The fluid is assumed to be viscous and incompressible with constant density ρ, gyration parameter j, and viscosity coefficients μ, $\varkappa$, α, β, γ.

b) We neglect any body forces acting on the fluid.

Under these assumptions, the equations of motion for the model are

$$\rho\frac{d\mathbf{q}}{dt}=-\operatorname{grad} p+\varkappa \operatorname{curl} \nu+(\mu+\varkappa)\Delta\mathbf{q}, \tag{1.a}$$

$$\operatorname{Div} \mathbf{q}=0, \tag{1.b}$$

$$\rho j\frac{d\nu}{dt}=-2\varkappa\nu+\varkappa \operatorname{curl} \nu-\gamma \operatorname{curl}\operatorname{curl} \nu+(\alpha+\beta+\gamma)\operatorname{grad}\operatorname{div} \nu, \tag{1.c}$$

$$\frac{dj}{dt}=0. \tag{1.d}$$

The material constants conform to the following inequalities

$$\mu\geqslant 0,\quad \varkappa\geqslant 0,\quad 3\alpha+\beta+\gamma\geqslant 0,\quad \gamma\geqslant 0,\quad |\beta|\leqslant\gamma.$$

2. Present Analysis. Let D be a fixed, bounded, three-dimensional region with smooth surface ∂D. The following boundary condition for (1.a)—(1.d)

$$\mathbf{q}=0, \quad \nu=0 \quad \text{on} \quad \partial D.$$

Physically, these boundary conditions appear to be reasonable.

Presently and throughout this study of the micropolar fluid model, we restrict our attention to smooth classical solutions which are assumed to exist [4].

Energy arguments will be employed much along the same lines as those used by Serrin [2] and Joseph [3].

The first result is the

Lemma 1. *We have,* $\forall t \geqslant 0$

$$\int_{D(t)} j \mathrm{d}x = \int_{D(0)} j \mathrm{d}x.$$

Proof. The result follows from a time-spatial integration (1.d) and then an application of the divergence theorem.

We now define the folowing function of time:

$$E(t) = \tfrac{1}{2} \int_{D(t)} \rho (q_i q_i + j \nu_i \nu_i) \mathrm{d}x.$$

Here the sumation convention is used. As one can readily see by its form, $E(t)$ is a measure of kinetic energy of micropolar fluid. Our next result states that this energy is bounded in time.

Lemma 2. *The energy $E(t)$ at any time is always less than or equal to the initial energy i.e.,* $\forall t \geqslant 0$

$$0 \leqslant E(t) \leqslant E(0).$$

Proof. Multiplying equation (1.a) by $\mathbf{q}$ and then integrating over D, we are lead to the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\tfrac{1}{2} \int_{D(t)} \rho q_i q_i \mathrm{d}x\right\} = -(\mu+\varkappa) \int_{D(t)} q_{i,j} q_{i,j} \mathrm{d}x + \varkappa \int_{D(t)} \nu . \mathrm{curl}\, \mathbf{q}\, \mathrm{d}x. \tag{2}$$

Similarly, multiplying equation (1.c) by ν and then integrating over D, we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left\{\tfrac{1}{2} \int_{D(t)} \rho j \nu_i \nu_i \mathrm{d}x\right\} = & -2\varkappa \int_{D(t)} \nu_i \nu_i \mathrm{d}x + \varkappa \int_{D(t)} \nu . \mathrm{curl}\, \mathbf{q}\, \mathrm{d}x - \gamma \int_{D(t)} \nu_{i,j} \nu_{i,j} \mathrm{d}x - \\ & -(\alpha+\beta) \int_{D(t)} (\mathrm{div}\, \nu)^2 \mathrm{d}x. \end{aligned} \tag{3}$$

Eqs. (2) and (3) imply

$$\dot{E}(t)=-\mu\int_{D(t)} q_{i,j}q_{i,j}\mathrm{d}x-\varkappa\int_{D(t)}(q_{i,j}-\varepsilon_{jik}\nu_k)(q_{i,j}-\varepsilon_{jis}\nu_s)\mathrm{d}x-$$

$$-\int_{D(t)}[\gamma\nu_{i,j}\nu_{i,j}+\alpha(\mathrm{div}\ \nu)^2+\beta\nu_{i,j}\nu_{j,i}]\mathrm{d}x, \tag{4}$$

i.e. $\mathrm{d}E/\mathrm{d}t\leqslant 0$. Integrating this differential inequality over the interval $[0, t]$ yields the desired result.

We are now in a position to demonstrate that the energy $E(t)$ actually decays to zero.

Theorem 1. *$E(t)$ decays to zero as t becomes infinitely large. That is, we have*

$$\operatorname*{Limit}_{t\to\infty} E(t)=0.$$

Proof. For the time derivative of $E(t)$ we have the expression (4). Integrating (4) over the interval $[0, t]$, we obtain

$$E(t)-E(0)=-\mu\int_0^t\int_{D(t)} q_{i,j}q_{i,j}\mathrm{d}x\mathrm{d}\tau-\varkappa\int_0^t\int_{D(\tau)}(q_{i,j}-\varepsilon_{jik}\nu_k)(q_{i,j}-\varepsilon_{jis}\nu_s)\mathrm{d}x\mathrm{d}\tau-$$

$$-\int_0^t\int_{D(\tau)}[\gamma\nu_{i,j}\nu_{i,j}+\alpha(\mathrm{div}\ \nu)^2+\beta\nu_{i,j}\nu_{j,i}]\mathrm{d}x\mathrm{d}\tau.$$

Since the last two terms of the above expression are nonpositive, it follows that

$$E(t)+\mu\int_0^t\int_{D(\tau)} q_{i,j}q_{i,j}\mathrm{d}x\mathrm{d}\tau\leqslant E(0).$$

Furthermore, since $E(t)$ is nonnegative, we conclude that $\forall t\geqslant 0$,

$$\mu\int_0^t\int_{D(\tau)} q_{i,j}q_{i,j}\mathrm{d}x\mathrm{d}\tau\leqslant E(0).$$

In particular

$$\mu\int_0^\infty\int_{D(\tau)} q_{i,j}q_{i,j}\mathrm{d}x\mathrm{d}\tau\leqslant E(0).$$

Since $\int_{D(t)} q_{i,j}q_{i,j}\mathrm{d}x$ is a continuous, positive function of t, then necessarily we must have

$$\operatorname*{Limit}_{t\to\infty}\int_{D(t)} q_{i,j}q_{i,j}\mathrm{d}x=0.$$

From a well-known integral inequality for functions with zero boundary data, we have

$$\int_{D(t)} q_i q_i \mathrm{d}x \leqslant \frac{1}{\lambda} \int_{D(t)} q_{i,j} q_{i,j} \mathrm{d}x, \tag{5}$$

where λ is a positive constant which depends only on geometry of ∂D. This inequality implies that

$$\operatorname*{Limit}_{t\to\infty} \int_{D(t)} q_i q_i \mathrm{d}x = 0. \tag{6}$$

As for the next term, we note that $\forall t \geqslant 0$

$$\varkappa \int_0^t \int_{D(\tau)} (q_{i,j} - \varepsilon_{jik} \nu_k)(q_{i,j} - \varepsilon_{jis} \nu_s) \mathrm{d}x \mathrm{d}\tau \leqslant E(0),$$

which implies that

$$\operatorname*{Limit}_{t\to\infty} \int_{D(t)} (q_{i,j} - \varepsilon_{jik} \nu_k)(q_{i,j} - \varepsilon_{jis} \nu_s) \mathrm{d}x = 0.$$

By use of the arithmetic-geometric mean inequality, it follows that for any $\alpha > 0$

$$\frac{\alpha - 1}{\alpha} q_{i,j} q_{i,j} + 2(1-\alpha) \nu_k \nu_k \leqslant (q_{i,j} - \varepsilon_{jik} \nu_k)(q_{i,j} - \varepsilon_{jis} \nu_s). \tag{7}$$

Choose $\alpha < 1$ so that $1 - 1/\alpha < 0$ and $1 - \alpha > 0$; hence, we have

$$0 \leqslant 2(1-\alpha) \int_{D(t)} \nu_k \nu_k \mathrm{d}x \leqslant \frac{1-\alpha}{\alpha} \int_{D(t)} q_{i,j} q_{i,j} \mathrm{d}x + \int_{D(t)} (q_{i,j} - \varepsilon_{jik} \nu_k)(q_{i,j} - \varepsilon_{jis} \nu_s) \mathrm{d}x. \tag{8}$$

Taking the limit as $t \to \infty$ in Eq. (8), we find

$$\operatorname*{Limit}_{t\to\infty} \int_{D(t)} \nu_k \nu_k \mathrm{d}x = 0. \tag{9}$$

Eqs. (6) and (9) give the desired result.

Actually, the proof of Theorem 1 affords information concerning the rate of decay of $E(t)$. It is not difficult to show that, in fact, $E(t)$ is of order $o(1/t)$.

It is known that the kinetic energy for the Navier-Stokes solution (with zero boundary data) decays exponentially in time, independent of the value of the constants ρ and μ. Hence, one would suspects that our energy $E(t)$ might also exhibit exponentially decay.

T h e o r e m 2. *There exists a positive constant η such that*

$$E(t) \leqslant E(t_0)[\exp - \eta(t - t_0)], \quad \forall t \geqslant t_0.$$

P r o o f. By using inequality (7) in Eq. (4), it follows that for any real, positive α

$$\dot{E}(t) \leqslant -\mu \int_{D(t)} q_{i,j} q_{i,j} \mathrm{d}x + \varkappa \frac{1-\alpha}{\alpha} \int_{D(t)} q_{i,j} q_{i,j} \mathrm{d}x + 2\varkappa(\alpha-1) \int_{D(t)} \nu_k \nu_k \mathrm{d}x -$$

$$- \int_{D(t)} [\gamma \nu_{i,j} \nu_{i,j} + \alpha(\operatorname{div} \nu)^2 + \beta \nu_{i,j} \nu_{j,i}] \mathrm{d}x.$$

Or

$$\dot{E}(t) \leqslant \left(-\mu + \varkappa \frac{1-\alpha}{\alpha}\right) \int_{D(t)} q_{i,j} q_{i,j} \mathrm{d}x + 2\varkappa(\alpha-1) \int_{D(t)} \nu_k \nu_k \mathrm{d}x.$$

Thus, with inequality (5) we obtain

$$\dot{E}(t) \leqslant -\lambda \left(\mu - \varkappa \frac{1-\alpha}{\alpha}\right) \int_{D(t)} q_i q_i \mathrm{d}x + 2\varkappa(\alpha-1) \int_{D(t)} \nu_k \nu_k \mathrm{d}x.$$

Now choose α such that

$$0 < \frac{\lambda}{\rho}\left(\mu - \varkappa \frac{1-\alpha}{\alpha}\right) = \frac{2\varkappa(1-\alpha)}{\rho j} = \frac{\eta}{2}. \tag{10}$$

Therefore, we obtain $\dot{E}(t) \leqslant -\eta E(t)$. Integrating this differential inequality over the time interval $[t_0, t]$, we find that $\forall t \geqslant t_0$

$$E(t) \leqslant E(t_0) \exp[-\eta(t-t_0)].$$

The theorem is proven.

Received December 5, 1984

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REFERENCES

1. Eringen A.C., J. Math. Mech., **16**, 1–8 (1966).
2. Serin J., Proc. Symp. Non-linear Problems, Univ. of Wisconsin, 1963, p. 69.
3. Joseph D., Arch. Ration. Mech. Anal., **22**, 163 (1966).
4. Sava V.Al., Rev. Roum. Math. Pures. Appl., **25**, 1 111–1 123 (1980).
5. Lakshmana Rao S.K., Qt. Appl. Math., **XXVI**, 278–280 (1969).

DESCREŞTEREA ENERGIEI CINETICE A FLUIDELOR MICROPOLARE INCOMPRESIBILE

(Rezumat)

Urmînd o cale diferită de cea folosită în [5] se arată că energia cinetică a fluidelor micropolare incompresibile descreşte la zero.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 532.5

THE STEADY CONE FLOW OF AN INCOMPRESSIBLE SIMPLE FLUID OF INTEGRAL TYPE

BY

C. FETECĂU and CORINA FETECĂU

1. Introduction. Numerous theoretical investigations have been reported, in the last time, about the problem of finding the motions of different non-Newtonian fluids filling some domains [1] ... [3]. In this paper we establish an exact solution for cone flow of an incompressible simple fluid of integral type of the first order.

The constitutive equation of a such fluid, as it was presented in [4] Sec. 37, is

$$\mathbf{T}=-p\mathbf{1}+\int_0^\infty \zeta(s)\mathbf{G}(s)\mathrm{d}s, \quad \mathbf{G}(.)=\mathbf{F}_t^t(.)^T\mathbf{F}_t^t(.)-\mathbf{1}, \tag{1.1}$$

where $\mathbf{T}$ is the stress tensor, p is the indeterminate pressure, $\mathbf{F}_t^t(.)$ is the relative deformation gradient history and $\zeta(.)\leqslant 0$ is a material function.

2. Statement of the Problem. We here investigate a flow whose velocity field, in the spherical coordinate system (r, θ, φ), has the contravariant components[1])

$$v^r=0, \quad v^\theta=0, \quad v^\varphi=\omega(r, \theta). \tag{2.1}$$

The path-lines of motion being ([4], Sec. **107**) the curves $\xi=\xi(\tau)$ defined by

$$\xi^1=r, \quad \xi^2=\theta, \quad \xi^3=\varphi+(\tau-t)\omega(r, \theta), \tag{2.2}$$

the physical components of $\mathbf{G}(.)$ are

$$\mathbf{G}(.)=\begin{pmatrix} \alpha^2(.) & \alpha(.)\beta(.) & \alpha(.) \\ \alpha(.)\beta(.) & \beta^2(.) & \beta(.) \\ \alpha(.) & \beta(.) & 0 \end{pmatrix}, \tag{2.3}$$

where $\alpha(s)=-rs\sin\theta\,\partial_r\omega$, $\beta(s)=-s\sin\theta\,\partial_\theta\omega$ and $s=t-\tau$.

When (1.1) and (2.3) are substituted into the equation of linear momentum

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[1] A more careful study of the integrability condition from [4] Sec. **115** or [5] § **21**, ensure us that a such motion with $\omega=\omega(\theta)$ does not exists.

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \tag{2.4}$$

where $\mathbf{b} = -\operatorname{grad} \nu$ denotes the body force per unit mass, one obtains the expression

$$p = -\rho\nu + \alpha\Phi + h(r, \theta, t) \tag{2.5}$$

for p, and the equation

$$\left(\partial_r^2 + \frac{4}{r}\partial_r\right)\omega + \frac{1}{r^2}(\partial_\theta^2 + 3 \operatorname{ctg} \theta\, \partial_\theta)\omega - \frac{a}{r^2 g \sin^2\theta} = 0 \tag{2.6}$$

for the unknown function ω.

Here $g = -\int_0^\infty s\zeta(s)ds$ and the constant a in (2.5) must vanish because p (and then T_{rr}, $T_{\theta\theta}$, $T_{\varphi\varphi}$) is a single-valued function of position. The other two equations from (2.4) can be used to determine the function h.

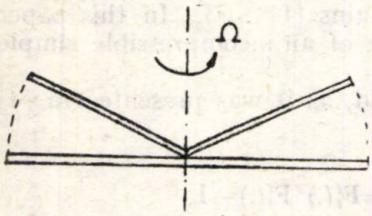

Fig. 1

We now suppose that the fluid adheres to a rigid plate, a rigid spherical disc of radius R and a rigid cone as shown in Fig. 1. The angle between plate and cone is denoted by α. We take the axis of the cone to be polar axis of the spherical coordinate system r, θ, φ, so that the equation of the cone is $\theta = \frac{\pi}{2} - \alpha$.

The cone is assumed to rotate about its axis with angular velocity Ω. The boundary conditions then are

$$\omega\left(r, \frac{\pi}{2}\right) = \omega(R, \theta) = 0, \quad \omega\left(r, \frac{\pi}{2} - \alpha\right) = \Omega. \tag{2.7}$$

3. Solution of the Problem. Denoting by $\bar{\omega} = \omega r^{3/2} \cos(\theta + \alpha)$ in (2.6) and taking into account (2.7) we reduce our problem to

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{9}{4r^2}\right)\bar{\omega} + \frac{1}{r^2}[\partial_\theta^2 + P(\theta)\partial_\theta + Q(\theta)]\bar{\omega} = 0 \tag{3.1}$$

and

$$\bar{\omega}(R, \theta) = \bar{\omega}\left(r, \frac{\pi}{2}\right) = \bar{\omega}\left(r, \frac{\pi}{2} - \alpha\right) = 0, \tag{3.2}$$

where

$$P(\theta) = 3 \operatorname{ctg} \theta + 2 \operatorname{tg}(\theta + \alpha),$$

$$Q(\theta) = 3 \operatorname{ctg} \theta \operatorname{tg}(\theta + \alpha) + \frac{1 + \sin^2(\alpha + \theta)}{\cos^2(\alpha + \theta)}.$$

If we now assume a separation of variables

$$\bar{\omega}(r, \theta) = U(r)V(\theta) \tag{3.3}$$

and substitute (3.3) in (3.1), we readily obtain

$$\frac{U''+\frac{1}{r}U'-\frac{9}{4r^2}U}{\frac{1}{r^2}U}=-\frac{V''+P(\theta)V'+Q(\theta)V}{V}=-\lambda, \tag{3.4}$$

where λ is a constant.

In view of (3.2) and (3.3), (3.4) reduces to

$$U''+\frac{1}{r}U'+\left(\lambda-\frac{9}{4}\right)\frac{1}{r^2}U=0, \quad U(R)=0 \tag{3.5}$$

and

$$V''+P(\theta)V'+Q(\theta)V=\lambda V, \quad V\left(\frac{\pi}{2}\right)=V\left(\frac{\pi}{2}-\alpha\right)=0. \tag{3.6}$$

The problem (3.6) admits, in general a denumerable infinity of real, positive and distinct eigenvalues ([6], Ch. X) $0<\lambda_1<\lambda_2<\lambda_3<\ldots\lambda_n<\ldots$ such that

$$\lim_{n\to\infty}\lambda_n=\infty. \tag{3.7}$$

The unique expressions for corresponding eigenfunctions $V_n(\theta)$ are obtained by introducing a normalisation condition

$$\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} V_n^2(\theta)\mathrm{d}\theta=1.$$

For the eigenvalues $\lambda_n\leqslant\frac{9}{4}$ only the solutions $U_n=0$ are bounded in $r=0$ and satisfy (3.5), while for eigenvalues $\lambda_n>\frac{9}{4}$ the general solution of this equation is

$$U_n=C_n[\cos(\mu_n\ln r)-\mathrm{ctg}(\mu_n\ln R)\sin(\mu_n\ln r)], \tag{3.8}$$

where $\mu_n=\sqrt{\lambda_n-\frac{9}{4}}$ and C_n are arbitrary constants. In view of these results we can write the solution of our problem under the form

$$\omega(r,\theta)=\frac{1}{r^{3/2}\cos(\theta+\alpha)}\sum_{n=1}^{\infty}C_nV_n(\theta)U_n(r). \tag{3.9}$$

As regards the constants C_n, we expect to be determined by experiments. The torque M which may be applied to produce the flow, the normal

stresses along the plate or the cone and the total force F necessary to keep cone and plate in place can be determined as in [4].

Received September 20, 1983

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REFERENCES

1. Fosdick R.L., Rajagopal K.R., Int. J. Nonlinear Mech., **13**, 131 (1978).
2. Rajagopal K.R., Gupta A.S., Int. J. Engng. Sci., **19**, 1401 (1981).
3. Rajagopal K.R., Gupta A.S., Int. J. Engng. Sci., **19**, 1009 (1981).
4. Truesdell C., Noll W., *Handbuch der Physik*. **III/3**, Springer Verlag, Berlin–Heidelberg–New York, 1965.
5. Coleman B.D., Markovitz H., Noll W., *Viscometric Flows of Non-Newtonian Fluids*. Springer Verlag, Berlin–Heidelberg–New York, 1966.
6. Tolstov G.P., *Serii Fourier* (transl. from russian). Ed. tehn., Buc., 1955.

CURGEREA CONICĂ STAŢIONARĂ A UNUI FLUID SIMPLU INCOMPRESIBIL DE TIP INTEGRAL

(Rezumat)

Se stabileşte o soluţie exactă pentru curgerea conică a unui fluid simplu incompresibil de tip integral, de ordinul întîi.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1–4, 1984
Secţia I
MECANICĂ

D.C. 532.5 ; 530.12

EXACT SOLUTIONS TO THE EQUATIONS OF RADIATIVE RELATIVISTIC PERFECT FLUIDS WITH CYLINDRICAL SYMMETRY

BY

GH. PROCOPIUC

1. Introduction. In a previous paper [1], we have studied the existence and uniqueness of spherically symmetric similarity solutions for the equations of radiative relativistic fluids. In this paper, we present exact solutions to the equations of radiative relativistic perfect fluids in a cylindrically symmetric space-time. Recently, an exact solution of the Einstein field equations for a viscous fluid which has its space-time geometry the Einstein–de Sitter metric was presented in [2].

2. Equations of Relativistic Radiative Perfect Fluids in a Cylindrically Symmetric Field. In comoving coordinates in which we choose units so that $c=1$, the line element of the space-time will be taken in the form

$$ds^2=e^{2\varphi}dt^2-e^{2\psi}dr^2-\rho^2 d\theta^2-dz^2, \tag{2.1}$$

where φ, ψ and ρ are functions of t and r, and the velocity vector is

$$u_\alpha=(e^\varphi, 0, 0, 0). \tag{2.2}$$

Since $q_\alpha u^\alpha=0$ and $q_\alpha q^\alpha=-q^2$, the radiative flux vector is

$$q_\alpha=(0, -qe^\psi, 0, 0). \tag{2.3}$$

Nontrivial components of the field equations

$$G_\alpha^\beta=\varkappa U_\alpha^\beta, \tag{2.4}$$

with $\varkappa=1$, where $G_\alpha^\beta=R_\alpha^\beta-\frac{1}{2}g_\alpha^\beta R$, and

$$U_\alpha^\beta=w^* u_\alpha u^\beta-p^* \overset{\perp}{g}{}_\alpha^\beta+u_\alpha q^\beta+q_\alpha u^\beta, \tag{2.5}$$

are

$$\frac{1}{\rho}\rho_t\psi_t e^{-2\varphi}-\left(\frac{1}{\rho}\rho_{rr}-\frac{1}{\rho}\rho_r\psi_r\right)e^{-2\psi}=-w^*, \tag{2.6}$$

$$\left(\frac{1}{\rho}\rho_{tt}-\frac{1}{\rho}\rho_t\varphi_t\right)e^{-2\varphi}-\frac{1}{\rho}\rho_r\varphi_r e^{-2\psi}=p^*, \tag{2.7}$$

$$(\psi_{tt}+\psi_t^2-\varphi_t\psi_t)e^{-2\varphi}-(\varphi_{rr}+\varphi_r^2-\varphi_r\psi_r)e^{-2\psi}=p^*, \tag{2.8}$$

$$\left(\frac{1}{\rho}\rho_{tt}+\psi_{tt}+\psi_t^2-\varphi_t\psi_t-\frac{1}{\rho}\rho_t\varphi_t+\frac{1}{\rho}\rho_t\psi_t\right)e^{-2\varphi}-$$
$$-\left(\frac{1}{\rho}\rho_{rr}+\varphi_{rr}+\varphi_r^2-\varphi_r\psi_r-\frac{1}{\rho}\rho_r\psi_r+\frac{1}{\rho}\rho_r\varphi_r\right)e^{-2\psi}=p^*, \tag{2.9}$$

$$\left(\frac{1}{\rho}\rho_{tr}-\frac{1}{\rho}\rho_t\varphi_r-\frac{1}{\rho}\rho_r\psi_t\right)e^{-(\varphi+\psi)}=-q. \tag{2.10}$$

The conservation conditions $U^{\beta}_{\alpha/\beta}=0$, become

$$w_t^*+\left(\frac{1}{\rho}\rho_t+\psi_t\right)(p^*+w^*)+\left[q_r+\left(\frac{1}{\rho}\rho_r+2\varphi_r\right)q\right]e^{\varphi-\psi}=0, \tag{2.11}$$

$$p_r^*+\varphi_r(p^*+w^*)+\left[q_t+\left(\frac{1}{\rho}\rho_t+2\psi_t\right)q\right]e^{-\varphi+\psi}=0. \tag{2.12}$$

The equations of radiation field $Q^{\beta}_{\alpha/\beta}=-F_\alpha$, with $F_\alpha=\sigma[q_\alpha+(Q-aT^4)u_\alpha]$, can be written as the pair

$$Q_t+\frac{4}{3}\left(\frac{1}{\rho}\rho_t+\psi_t\right)Q+\left[q_r+\left(\frac{1}{\rho}\rho_r+2\varphi_r\right)q\right]e^{\varphi-\psi}=-\sigma(Q-aT^4)e^{\varphi}, \tag{2.13}$$

$$\frac{1}{3}Q_r+\frac{4}{3}\varphi_rQ+\left[q_t+\left(\frac{1}{\rho}\rho_t+2\psi_t\right)q\right]e^{-\varphi+\psi}=-\sigma qe^{\psi}. \tag{2.14}$$

In the above equations, $w^*=w+Q$ and $p^*=p+\frac{1}{3}Q$ are the total energy and the total pressure of fluid.

From (2.6)—(2.7) and (2.8)—(2.9) it results straightforward $w^*=p^*$, and then equation (2.9) is not independent of the others.

Eqs. (2.6)—(2.7), with the help of (2.10), are conveniently transformed into

$$(\rho_t^2e^{-2\varphi}-\rho_r^2e^{-2\psi})_r=-2\rho(\rho_rp^*+q\rho_te^{-\varphi+\psi}), \tag{2.15}$$

$$(\rho_t^2e^{-2\varphi}-\rho_r^2e^{-2\psi})_t=-2\rho(\rho_tp^*+q\rho_te^{-\varphi-\psi}), \tag{2.16}$$

The system (2.15)—(2.16) is integrable. To proof this is sufficient to prove that

$$(\rho(\rho_rp^*+q\rho_te^{-\varphi+\psi}))_t+(\rho(\rho_tp^*+q\rho_re^{\varphi-\psi}))_r=0.$$

Note that this condition is ensured by (2.11) and (2.12), taking into account (2.10).

It follows that there exists a function $m(t, r)$ such that

$$m_r=-2\rho(\rho_rp^*+q\rho_te^{-\varphi+\psi}), \tag{2.17}$$

$$m_t=2\rho(\rho_tp^*+q\rho_re^{\varphi-\psi}). \tag{2.18}$$

If we return to (2.15)—(2.16), we have

$$(\rho_t^2 e^{-2\varphi}-\rho_r^2 e^{-2\psi})_r=m_r, \qquad (\rho_t^2 e^{-2\varphi}-\rho_r^2 e^{-2\psi})_t=m_t.$$

From these equations we obtain up to an additive constant

$$\rho_t^2 e^{-2\varphi}-\rho_r^2 e^{-2\psi}=m. \tag{2.19}$$

3. Exact Solutions with Cylindrical Symmetry. We are concerned with similarity solutions defined as in [1].

Putting $v=r/t$, for a similarity solution we have

$$\varphi(t,r)=\Phi(v), \quad \psi(t,r)=\Psi(v), \quad \rho(t,r)=rR(v). \tag{3.1}$$

Furthermore, from the Eqs. (2.19), (2.10), (2.8) it results that m, r^2p^*, r^2q are functions only of v. Therefore, we can denote

$$m=M(v), \quad \rho^2p^*=R^2r^2p^*=P(v), \quad \rho^2q=R^2r^2q=K(v) \tag{3.2}$$

and

$$\rho^2Q=\rho K_1(v)+K_2(v)=rR(v)K_1(v)+K_2(v). \tag{3.3}$$

Next, for a function $f=f(v)$, we denote $vf'(v)=\dot{f}$; then $v^2f''(v)=\ddot{f}-\dot{f}$.

Consequently, the Eqs. (2.19), (2.8), (2.10) can be rewritten in the forms

$$M=v^2\dot{R}^2e^{-2\Phi}-(R+\dot{R})^2e^{-2\Psi}, \tag{3.4}$$

$$P=v^2R^2(\ddot{\Psi}+\dot{\Psi}+\dot{\Psi}^2-\dot{\Phi}\dot{\Psi})e^{-2\Phi}-R^2(\ddot{\Phi}-\dot{\Phi}+\dot{\Phi}^2-\dot{\Phi}\dot{\Psi})e^{-2\Psi}, \tag{3.5}$$

$$K=vR(\ddot{R}+\dot{R}-\dot{\Phi}\dot{R}-\dot{\Psi}(R+\dot{R}))e^{-\Phi-\Psi}. \tag{3.6}$$

From (2.17)—(2.18) we get, with $V=ve^{-\Phi+\Psi}$,

$$\dot{M}=-2\left(\frac{R+\dot{R}}{R}P-\frac{\dot{R}}{R}VK\right)=2\left(\frac{\dot{R}}{R}P-\frac{R+\dot{R}}{R}\frac{K}{V}\right), \tag{3.7}$$

and from (2.11)—(2.12)

$$\dot{P}+2\dot{\Psi}P=\frac{1}{V}\left(\dot{K}-\frac{R+\dot{R}}{R}K+2\dot{\Phi}K\right), \tag{3.8}$$

$$\dot{P}-2\frac{R+\dot{R}}{R}P+2\dot{\Phi}P=V\left(\dot{K}-\frac{\dot{R}}{R}K+2\dot{\Psi}K\right). \tag{3.9}$$

To satisfy (2.14) for any r we must take

$$\dot{K}_1-\frac{R+\dot{R}}{R}K_1+4\dot{\Phi}K_1=-3\sigma\frac{K}{R}e^{\Psi}, \tag{3.10}$$

$$\dot{K}_2-2\frac{R+\dot{R}}{R}K_2+4\dot{\Phi}K_2=3V\left(\dot{K}-\frac{\dot{R}}{R}K+2\dot{\Psi}K\right). \tag{3.11}$$

Now we shall seek similarity solutions for which $R(v)=1$. Hence M, K, P will be given by

$$M=-e^{-2\Psi}, \tag{3.4'}$$

$$P=v^2(\ddot{\Psi}+\dot{\Psi}+\dot{\Psi}^2-\dot{\Phi}\dot{\Psi})e^{-2\Phi}-(\ddot{\Phi}+\dot{\Phi}^2-\dot{\Phi}-\dot{\Phi}\dot{\Psi})e^{-2\Psi}, \tag{3.5'}$$

$$K=-v\dot{\Psi}e^{-\Phi-\Psi}; \tag{3.6'}$$

then

$$\dot{M}=-2P=-2\frac{K}{V}, \tag{3.7'}$$

$$\dot{P}+2\dot{\Psi}P=\frac{1}{V}(\dot{K}-K+2\dot{\Phi}K), \tag{3.8'}$$

$$\dot{P}-2P+2\dot{\Phi}P=V(K+2\dot{\Psi}K), \tag{3.9'}$$

$$\dot{K}_1-K_1+4\dot{\Phi}K_1=-3\sigma Ke^{\Psi}, \tag{3.10'}$$

$$\dot{K}_2-2K_2+4\dot{\Phi}K_2=3V(\dot{K}+2\dot{\Psi}K). \tag{3.11'}$$

Eq. (2.13) gives the temperature T.

From (3.8′) and (3.9′) substituting P and K given by (3.6′) and (3.7′) we get for Φ and Ψ the equations

$$\dot{\Phi}=\dot{\Psi}, \tag{3.12}$$

$$\ddot{\Psi}-2\dot{\Psi}-2\dot{\Psi}^2+2\dot{\Phi}\dot{\Psi}=V^2(\ddot{\Psi}+\dot{\Psi}+\dot{\Psi}^2-\dot{\Phi}\dot{\Psi}). \tag{3.13}$$

Eq. (3.5′), in virtue of (3.12) and (3 13), is identically satisfied.

System (3.12)—(3.13) can be integrated to yield

$$\Phi(v)=\Psi(v)+\ln\alpha, \qquad \alpha>0,$$

$$\Psi(v)=\begin{cases}\Psi_\infty-\alpha K_\infty(\alpha^2-v^2)^{-\frac{1}{2}}, & v\in[0,\alpha),\\ \Psi_\infty+\alpha K_\infty(v^2-\alpha^2)^{-\frac{1}{2}}, & v\in(\alpha,+\infty),\end{cases}$$

so that

$$P(v)=\begin{cases}\alpha K_\infty v^2(\alpha^2-v^2)^{-\frac{3}{2}}\exp(-2(\Psi_\infty-\alpha K_\infty(\alpha^2-v^2)^{-\frac{1}{2}})), & v\in[0,\alpha),\\ \alpha K_\infty v^2(v^2-\alpha^2)^{-\frac{3}{2}}\exp(-2(\Psi_\infty+\alpha K_\infty(v^2-\alpha^2)^{-\frac{1}{2}})), & v\in(\alpha,+\infty),\end{cases}$$

$$K(v)=\alpha vP(v),$$

where $\alpha=\exp(\Phi_\infty-\Psi_\infty)$, and Φ_∞, Ψ_∞, K_∞ are the values of Φ, Ψ, K, respectivelly, for $v\to\infty$.

Substituting now Φ, Ψ and K into (3.10′) and (3.11′), we obtain by integration

$$\zeta_1(v)=\begin{cases} K_1^0 v\exp(4\alpha K_\infty(\alpha^2-v^2)^{-\frac{1}{2}})+\dfrac{\sigma}{\alpha}v\exp(-\Psi_\infty+\alpha K_\infty(\alpha^2-v^2)^{-\frac{1}{2}}), & v\in[0,\ \alpha), \\ K_1^0 v\exp(-4\alpha K_\infty(v^2-\alpha^2)^{-\frac{1}{2}})+\dfrac{\sigma}{\alpha}v\exp(-\Psi_\infty-\alpha K_\infty(v^2-\alpha^2)^{-\frac{1}{2}}), & v\in(\alpha,\ +\infty); \end{cases}$$

$$\zeta_2(v)=\begin{cases} K_2^0 v^2\exp(4\sigma K_\infty(\alpha^2-v^2)^{-\frac{1}{2}})-\dfrac{9v^2}{4\alpha^2K_\infty^2}(2\alpha^2K_\infty^2(\alpha^2-v^2)^{-1}+ \\ \qquad +2\alpha K_\infty(\alpha^2-v^2)^{-\frac{1}{2}}+1)\exp(-2\Psi_\infty+2\alpha K_\infty(\alpha^2-v^2)^{-\frac{1}{2}}), & v\in[0,\alpha), \\ K_2^0 v^2\exp(-4\alpha K_\infty(v^2-\alpha^2)^{-\frac{1}{2}})+\dfrac{9v^2}{4\alpha^2K_\infty^2}(2\alpha^2K_\infty^2(v^2-\alpha^2)^{-1}- \\ \qquad -2\alpha K_\infty(v^2-\alpha^2)^{-\frac{1}{2}}+1)\exp(-2\Psi_\infty+2\alpha K_\infty(v^2-\alpha^2)^{-\frac{1}{2}}), & v\in(\alpha,\ +\infty), \end{cases}$$

where K_1^0 and K_2^0 are constants.

Received December 5, 1984

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REFERENCES

1. Procopiuc Gh., *Spherically Symmetric Similarity Solutions of the Equations of Radiative Relativistic Fluids*. Rev. Roum. Sci. Techn., Méc. Appl., 27, 6, 725–736 (1982).
2. Coley A.A., Tupper B.O.J., *An Exact Viscous Fluid FRW Cosmology*. Phys. Letters, 95 A, 7, 357–360 (1983).
3. Bressan A., *Relativistic Theories of Materials*. Springer Tracts in Natural Philosophy, Vol. 29, Springer Verlag, Berlin–Heidelberg–New York, 1978.

SOLUŢII EXACTE ALE ECUAŢIILOR FLUIDELOR PERFECTE RELATIVISTE RADIATIVE CU SIMETRIE CILINDRICĂ

(Rezumat)

Se consideră un fluid perfect în prezenţa unui cîmp de radiaţie termică într-un spaţiu-timp raportat la un sistem de coordonate cilindrice. Presupunînd coeficienţii metricii funcţii numai de t şi r, se integrează sistemul format din ecuaţiile lui Einstein, ecuaţiile hidrodinamicii şi ecuaţiile cîmpului de radiaţie.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 533

NONLINEAR PLASMA DYNAMICS OF THE ELECTROSTATIC DRIFT WAVE TURBULENCE

BY

CORNELIU CIUBOTARIU and VIOREL STANCU

1. Introduction. During the past few years a number of papers have appeared which discuss the nonlinear drift waves in connection with enhanced diffusion in magnetic confinement devices. Drift wave fluctuations which evolve into a weak turbulence state [1] ... [3] or into a strongly turbulent state [4], [5] have received considerable interest, and are investigated both theoretically and experimentally. The recent observations on turbulence in tokamaks indicate that the plasma fluctuations lead to a strongly nonlinear state and emphasize the need for a complete nonlinear theory of drift waves for evaluating their effects on plasma transport [6].

There are three approaches for the study of the drift wave turbulence. The first method (see, for example, [7]) applies the concept of a conventional perturbation theory (a multiple scale expansion in time) directly to the fluid model equations for the magnetized ion -electron plasma. The second method [8] also uses the fluid equations but in their general form which includes the electron thermal effects in the equations of motion and the non-zero unperturbed mean velocities. In this method, the interaction of drift waves with frequencies much smaller than the ion gyrofrequency appears as a particular example of a normal mode analysis. In the third approach [9] ... [12], other authors found it advantageous to reduce and simplify the plasma equations before attempting a spectral description of the turbulence.

The direction of energy and enstrophy flows in the wavenumber space is of key interest for several of the proposed theories. This direction may be found by studying the coupling coefficients for three-wave interaction of drift waves in a non-uniform magnetized plasma. Although the inhomogenity is the main characteristic of the plasma which leads to linear drift waves, the influence of this on the mode coupling process is neglected in the last two methods indicated above. The neglecting of the inhomogenity of wave number does not permit to distinguish between the nonlinear mode-coupling terms in the cases of uniform and nonuniform plasma. In the first method the spatially varying wave number is taken into consideration but the three particular waves considered by K a t o u [7] to argue the isotropization of the drift-wave spectrum in the plane perpendicular to the mag-

netic field, do not satisfy the conditions implied by the nonuniformity of the plasma : $k_{\alpha x}=k_{\alpha x}(x)$, $k_{\alpha y}=\text{const.}$, $\alpha=1, 2, 3$.

In this Note, we show that in the three-wave interaction process, the inhomogenity of wave number can be most important and the new corresponding coupling terms can even dominate the coupling coefficients for the case of constant wave number. We investigate the weak electrostatic drift wave turbulence on the basis of WKB analysis.

2. Generalization of the Hasegawa-Mima Equation. Many properties of the nonlinear behaviour of low frequency waves in plasma can be described by the set of quadratically coupled equations

$$\frac{d}{dt}\varphi_{\vec{k}_\alpha}(t)+i\omega_{\vec{k}_\alpha}\varphi_{\vec{k}_\alpha}(t)=\sum_{\vec{k}_1+\vec{k}_2+\vec{k}_3=0} L_{\vec{k}_\alpha\vec{k}_\beta\vec{k}_\gamma}\varphi^*_{\vec{k}_\beta}(t)\varphi^*_{\vec{k}_\gamma}(t) \tag{2.1}$$

for certain $\omega_{\vec{k}_\alpha}$ and $L_{\vec{k}_\alpha\vec{k}_\beta\vec{k}_\gamma}$. In the case of electrostatic drift wave turbulence, $\varphi_{\vec{k}_\alpha}$ is the fluctuating electric potential and $\omega_{\vec{k}_\alpha}$ is the drift wave frequency. The Eqs. (2.1) can be obtained if we substitute a spatial Fourier series,

$$\varphi(\vec{x}, t)=\sum_{\vec{k}}\varphi_{\vec{k}}(t)\exp(i\vec{k}.\vec{x})+\text{c.c.}, \tag{2.2}$$

into the ion-vortex equation of H a s e g a w a and M i m a [9]

$$\frac{\partial}{\partial t}(\nabla^2\varphi-\varphi)-[(\nabla\varphi\times\hat{z}).\nabla]\left(V^2\varphi+\ln\frac{\omega_{ci}}{n_0}\right)=0. \tag{2.3}$$

In order to compare our results with those of K a t o u [7] we first generalize Eq. (2.3) to the case when the condition of quasineutrality is not used, that is

$$n=n_0(x)(1+\varphi)-\frac{c_A^2}{c^2}\nabla^2\varphi. \tag{2.4}$$

Here, φ is the electrostatic potential normalized by $\frac{T_e}{e}$, $\frac{c_A^2}{c^2}=\frac{B_0^2}{4\pi m_i n_0 c^2}$ and $T_e(\gg T_i)$ is the electron temperature. The plasma whose density n_0 in the absence of the drift waves varies with x is immersed in a uniform magnetic field $\vec{B}=B_0\hat{z}$. We assume that during the propagation of drift waves, the small electron inertia allows a rapid motion of electrons in the direction of the ambient magnetic field (adiabatic electrons [13]) and leads to a electron Boltzmann distribution. If we non-dimensionalize the equations, we obtain the expressions of the drift velocity

$$\vec{v}_\perp=-\nabla\varphi\times z, \tag{2.5}$$

the vorticity

$$\vec{\Omega}=\nabla\times\vec{v}_\perp=\nabla^2\varphi\hat{z}, \tag{2.6}$$

and a generalized form of the Hasegawa-Mima equation

$$\left(\frac{\partial}{\partial t}+\vec{v}_\perp.\nabla\right)\nabla^2\varphi-\frac{1}{n_0}\left(\frac{\partial}{\partial t}+\vec{v}_\perp.\nabla\right)\left(n_0+n_0\varphi-\frac{c_A^2}{c^2}\nabla^2\varphi\right)=0. \tag{2.7}$$

The dynamic change of the spatial Fourier spectrum of this equation is analogous to that obtained from Eqs. (2.1) and (2.3) where

$$L_{\vec{k}_\alpha \vec{k}_\beta \vec{k}_\gamma} = \frac{1}{1+k_\alpha^2} (\vec{k}_\beta \times \vec{k}_\gamma) . \hat{z}(k_\gamma^2 - k_\beta^2) \delta_{\vec{k}_\alpha + \vec{k}_\beta + \vec{k}_\gamma, 0} \tag{2.8}$$

and

$$\delta_{\vec{k}_\alpha + \vec{k}_\beta + \vec{k}_\gamma, 0} = \begin{cases} 1 & \text{if } \vec{k}_\alpha + \vec{k}_\beta + \vec{k}_\gamma = 0, \\ 0 & \text{if } \vec{k}_\alpha + \vec{k}_\beta + \vec{k}_\gamma \neq 0. \end{cases} \tag{2.9}$$

3. Coupling Coefficients with Spatially Varying Wave Number. To find the turbulence properties of Eq. (2.7), we consider three plane waves with wave vectors $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_3$, and frequencies ω_1, ω_2 and ω_3. Since we look for a temporally (absolute) growing instability, we assume that the wave amplitudes have a small temporal variation due to the nonlinear interaction between the waves and a small spatial change due to the linear WKB effects, which is proportional to the factor $k_\varkappa^{-1/2}$, i.e.

$$\varphi(\vec{x}, t) = \sum_{\alpha=1}^{3} \varphi_{\vec{k}_\alpha}(t) \frac{1}{\sqrt{k_{\alpha x}}} \exp\left[i \left(\int k_{\alpha x} dx + k_{\alpha y} dy \right) \right] + \text{c.c.} \tag{3.1}$$

In an inhomogeneous plasma, the wave numbers of the three waves generally have different spatial variation, and therefore the matching condition holds only in a limited region in the direction of the inhomogenity. However, if the density gradient is exponential, $\nabla \ln n_0$ is constant and thus the k matching can be satisfied exactly. If the plasma has a general monotonic density profile, the variation of the drift wave frequency around the point of exact resonance can be interpreted as an equivalent wave-number shift or mismatch given by [14]

$$k_{1x}(\Delta L) + k_{2x}(\Delta L) + k_{3x}(\Delta L) \cong \frac{\omega_{p0}^2}{2k_0^2 c^2} \frac{1}{n_0} \frac{dn_0}{dx} \Delta L, \tag{3.2}$$

where ω_{p0}, n_0 and dn_0/dx denote plasma frequency, density, and density gradient respectively at the resonance point and ΔL is the distance from this point. However, in the mode-coupling equations we can ignore the phase factor $\exp\left[-i \int dx \sum_{\alpha=1}^{3} k_{\alpha x}(x) \right]$, since we shall assume a large frequency mismatch.

If we consider the orderings corresponding to a fusion device [6]

$$\begin{gathered} \varepsilon = 10^{-2}, \quad \varphi \sim \varepsilon, \quad \omega \sim \varepsilon, \quad \varkappa \equiv -\frac{1}{2k_x} \frac{dk_x}{dx} \sim \varepsilon^{1/2}, \\ v_d \equiv -\frac{1}{n_0} \frac{dn_0}{dx} \sim \varepsilon^{1/2}, \quad \frac{c_A^2}{c^2} \sim \varepsilon^{3/2}, \quad \frac{\partial}{\partial t} \sim \varepsilon^{1/2}, \\ k_x \rho \sim k_y \rho_s \sim 1, \quad \rho_s = \frac{1}{\omega_{ci}} \left(\frac{T_e}{m_i} \right)^{1/2}, \end{gathered} \tag{3.3}$$

the nonlinear mode-coupling equations obtained from Eq. (2.7) when we only keep resonant terms of the second order are

$$\frac{d\varphi_\alpha}{dt}+i\omega_\alpha\varphi_\alpha=\Lambda_{\alpha\beta\gamma}\varphi_\beta^*\varphi_\gamma^*, \tag{3.4}$$

where

$$\omega_\alpha=\omega_{\vec{k}_\alpha}=-\frac{\vec{k}_\alpha\times\hat{z}\,.\,\nabla\ln n_0}{1+k_\alpha^2 f^2},\qquad \varphi_\alpha\equiv\varphi_{\vec{k}_\alpha}(t), \tag{3.5}$$

$$f^2=1+\frac{c_A^2}{n_0c^2},\quad \alpha,\beta,\gamma=1,2,3,$$

and

$$\Lambda_{\alpha\beta\gamma}=\frac{f^2}{1+k_\alpha^2 f^2}\Big\{(\vec{k}_\beta\times\vec{k}_\gamma).\hat{z}(k_\gamma^2-k_\beta^2)- \tag{3.6}$$

$$-i\Big[(k_\gamma^2-k_\beta^2)(k_{\beta y}\varkappa_\gamma-k_{\gamma y}\varkappa_\beta)+2\Big(k_{\gamma y}k_{\beta x}\frac{dk_{\beta x}}{dx}+k_{\beta y}k_{\gamma x}\frac{dk_{\gamma x}}{dx}\Big)\Big]\Big\}.$$

The first term on the right hand side of Eq. (3.6) arises from the $\vec{E}\times\vec{B}$ drift and is identical with the corresponding terms from the previous works [6] ... [8], [9], [11]. The second term appears from the spatial nonuniformity of the unperturbed plasma density. We see that even in our pseudo-three-dimensional formulation ($k_{\|}\equiv k_z\simeq 0$) the three-wave interaction, described by the coupling coefficients with spatially varying wave number Λ, makes the electrostatic fluctuation spectrum isotropic in the wave number plane perpendicular to the magnetic field, as is the case in the three-dimensional wave dynamics developed by K a t o u [7].

It is known that the nonlinear wave interactions tend to be strongest when the wave vectors are collinear. In this case or in a less particular case when the wave vectors are almost parallel to each other, the additional terms which appear in comparison of Eq. (3.6) with Eq. (2.8) can be most important and even dominate the whole coupling coefficient.

4. Is it Possible to Obtain a General Coupling Coefficient? Recently, L i n d g r e n, L a r s s o n and S t e n f l o [8] have calculated the local coupling coefficients in order to include wave propagation in any kind of non-uniform plasma which can be described by the standard fluid equations with an isotropic pressure term, but in the particular case of the interaction of three drift waves, the Fourier analysis yields the interaction kernel (2.8) of Hasegawa and Mima which does not contain terms refering to the nonuniformity of the plasma. For the pseudo-three-dimensional turbulence, a local coupling coefficient is

$$V_3=i\frac{n_0m_i}{\omega_3B_0^4}\left[(\nabla\varphi_1\times\nabla\varphi_3^*).\vec{B}_0\nabla^2\varphi_1+(\nabla\varphi_1\times\nabla\varphi_3^*).\vec{B}_0\nabla^2\varphi\right] \tag{4.1}$$

and satisfies a coupled equation of the form

$$\frac{\partial a_3}{\partial t}+i\omega_3a_3=-i\omega_3V_3(C_3^*)^{-1}\exp(-i\omega_3t), \tag{4.2}$$

where a_3 is a normal mode and C_3 is a slowly varying amplitude.

Within a WKB analysis, using Eq. (3.1) for three interacting waves with $\vec{k}_1+\vec{k}_2=\vec{k}_3$ and $\omega_1+\omega_2=\omega_3$, we obtain

$$V_3=-i\frac{n_0 m_i}{\omega_3 B_0^3}[(\vec{k}_1\times\vec{k}_2).\hat{z}(k_2^2-k_1^2)- \\ -ik_2^2(\varkappa_1 k_{3y}+\varkappa_3 k_{1y})-ik_1^2(\varkappa_2 k_{3y}+\varkappa_3 k_{2y})]\varphi_1\varphi_2\varphi_3^*. \tag{4.3}$$

We see that the terms which take into consideration the inhomogenity of the wave number have not been obtained in the previous work. The drift wave absolute instability of the inhomogeneous plasma and thus the transition of the nonlinear oscillations to turbulence is decided by the sign of the product of the form $\Lambda_{\alpha\beta\gamma}\Lambda_{\beta\gamma\alpha}$. This means that the k-spectrum will cascade in a different way with respect to the classical dual cascade of the two-dimensional Navier-Stokes or Hasegawa-Mima equations. The final conclusion is that in the case of electrostatic drift wave turbulence, it is not possible to obtain the general coupling coefficients which contain all physical quantities implied in the description of turbulent states. The present result differs from earlier results in which the density gradient is taken as exponential thereby the mode coupling equations become homogeneous.

Received December 14, 1984

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REFERENCES

1. Tsytovich V.N., *Nonlinear Effects in Plasma*. Plenum, New York, 1970.
2. Sagdeev R.Z., Galeev A.A., *Nonlinear Plasma Theory*. Benjamin, New York, 1969.
3. Davidson R.C., *Methods in Nonlinear Plasma Theory*. Academic Press, New York, 1972.
4. Chen F.F., Phys. Rev. Lett., **15**, 381 (1965).
5. Pécseli H.L., Physica Scripta, **T2**, 147 (1982).
6. Mazzucato E., Phys. Rev. Lett., **36**, 792 (1976).
7. Katou K., J. Phys. Soc. Japan, **51**, 996 (1982).
8. Lindgren T., Larsson J., Stenflo L., J. Plasma Phys., **26**, 407 (1981).
9. Hasegawa A., Mima K., Phys. Rev. Lett., **39**, 205 (1977).
10. Hasegawa A., Mima K., Phys. Fluids, **21**, 87 (1978).
11. Hasegawa A., Kodama Y., Phys. Rev. Lett., **41**, 1 470 (1978).
12. Horton W., Phys. Rev. Lett., **37**, 1 269 (1976).
13. Terry P.W., Horton W., IFS Report 56, Austin, Texas, 1982.
14. Anderson D., Wilhelmsson H., Physica Scripta, **11**, 341 (1975).

DINAMICA DE PLASMĂ NELINIARĂ A TURBULENŢEI UNDELOR DE DRIFT ELECTROSTATIC

(Rezumat)

Folosind analiza WKB a ecuaţiilor care descriu turbulenţa undelor de drift electrostatic, se obţin coeficienţii de cuplaj pentru cazul cînd neuniformitatea plasmei implică o variaţie spaţială a numerelor de undă. Se arată că nu pot fi formulate expresii generale pentru aceşti coeficienţi de cuplaj a modurilor, deoarece stările de turbulenţă din plasmă conduc la o mare diversitate de cascade de spectre în funcţie de condiţiile iniţiale.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 530.12

DEVICE FOR ADJUSTMENT IN VACUUM OF THE POSITION OF A TORSION PENDULUM FOR VERIFYING THE EQUIVALENCE PRINCIPLE

BY

RODICA MARIA NEAGU, I. CONDREA and CARMINA IGNAT

1. Introduction. The inertial forces are proportional to the masses of particles and, other conditions being equal, impart the same relative accelerations to the particles.

Gravitational forces have the same property: at the same point in a gravitational field, these forces, like inertial forces are proportional to the masses of the particles and impart the same acceleration, equal to the field intensity, to all of them. Consequently free motion of a body with respect to a noninertial frame of reference is equivalent to its motion with respect to an inertial frame accomplished under the action of a certain additional, equivalent, gravitational field. This statement is called the e q u i v a l e n c e p r i n c i p l e.

Mass appearing in the equation that expresses Newton's second law represents the inertial properties of a particle and is called its inert mass. The mass of a particle appearing in the equation that expresses the law of gravitation, represents the gravitational properties of the particle and is called its gravitating mass.

It has been found on the basis of extremely precise experiments that the ratio of their inert mass to their gravitating mass is the same for all bodies.

2. General Data about the Torsion Balance of Braginski Type. The equivalence between the inert mass and its gravitating mass has been verified with a relative precision of 10^{-11} in the experiment done by D i c k e—R o l l—K r o t k o v [1]. In the experiment later achieved by V. B r a g i n s k i and V. P a n o v [2] it was proved that the ratio between the inert mass and the gravitating mass for Al and Pt are the same with a relative precision of $0.9\ 10^{-12}$, with a certitude coefficient of 0,95, in conformity with the Student criteria.

In the Polytechnic Institute of Jassy, Department of Physics [3], a torsion balance of Braginski type has been made with the purpose of verifying the equivalence principle.

In the following we shall give several general data on the balance built in Polytechnic Institute of Jassy (in a initial stage) and on the torsion balance of Braginski type.

The torsion balance of Braginski type is a mechanical oscillator with a great relaxation time ($\sim 10^7$s). The balance built in an initial stage in Polytechnic Institute of Jassy has a relaxation time about $5\ 10^2$ s.

In order to point out deviation from the equivalence principle with a precision of 10^{-12} it is necessary to be able to estimate a couple of forces equal to $8\ 10^{-19}$N.m and an amplitude of the angular oscillations of the torsion pendulum about $1.8\ 10^{-7}$ radians. The balance of Braginski and Panov had the following characteristics : total mass of the arm balance and of the test bodies, 4.7 g ; lenght of the body containing the balance, 3.5 m ; lenght of the torsion wire of wolfram of $5\ 10^{-6}$ m diameter, 2.8 m ; pression inside the body containing the balance, about 10^{-9} torr. The minimal acceleration of a test mass m, which can be distinguished on the background of thermical fluctuations, with the Braginski balance, in the classical approximation, is about 10^{-13} cm s^{-2}.

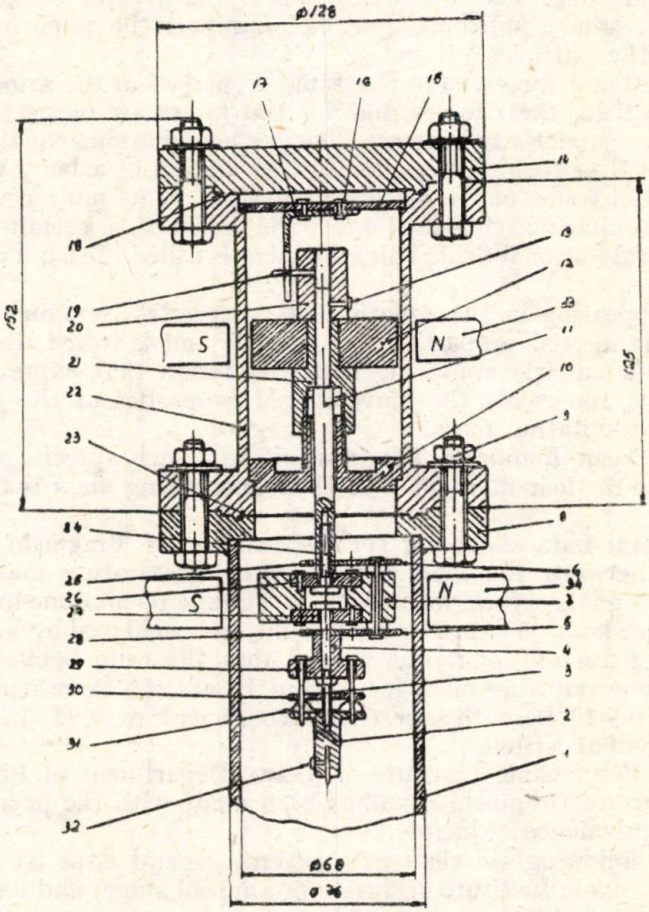

Fig. 1

The balance built in Polytechnic Institute of Jassy has the following characteristics, in the actual experimental pattern without vacuum : mass of the mobile equipment, 4.6728 g; torsion wire of nickeline or constantan with a diameter of 5 10^{-5} m ; lenght of the wire about 3 m. The minimal acceleration that can be distinguished is 6.89 10^{-10} cm s^{-2}.

3. Description of the Adjustment Device in Vacuum of the Torsion Pendulum Position. In the upper part of the installation achieved, we have fixed the rotation and translation device made of stainless steel and soft iron. This devices allows us to fix the equilibrium position of the pendulum which is found inside the vacuum room. Our device is of the Roll-Krotkov-Dicke type. By means of this device we can adjust the torsion balance with the purpose of acquiring a correct position of the optical system.

The adjustment devices in vacuum of the torsion pendulum position allows us to rotate and to vertically translate the wire independently (Fig. 1). The pendulum is fixed in the piece *2*. The vertical translation is accomplished by means of the rod *11*, fixed by the screw *13* to the piece *20*. This is supported by the soft iron piece *12*, which is joint with piece *21* with an interior thread. In this piece we have screwed the support *10*. It has an exterior thread *16* and is fixed on the cylindical wall *22*, which limits the vacuum room.

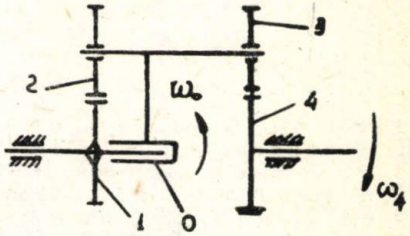

Fig. 2

The piece *20* is prevented to rotate by the fork pin *19* which penetrates in the fork *17*. By the rotation of the subdevices made up of the pieces *12—21*, the latter rise or falls and thus the piece *20*, together with the pendulum, has only a vertical movement. For a complete rotation of the soft iron piece *12* the vertical translation is *2* mm. In the actual form the maximal vertical movement is 15 mm. This distance can be modified by changing the lenght of the threads from the pieces *21* and *10*.

The rotation mechanism of the pendulum is placed under the support *10*; it is a epicycloidal differential gearing. The differential spider is formed by the soft iron piece *7*. The diagram of the differential mechanism is given in Fig. 2. The four wheels have the following number of teeth : $z_1=39$; $z_2=39$; $z_3=38$; $z_4=40$. Because $\omega_1=0$, it results

$$\omega_4=\omega_0\left(1-\frac{z_1z_3}{z_2z_4}\right)=\frac{\omega_0}{20}.$$

The demultiplication ratio is 1 : 20. This ratio can be changed by replacing the set of four wheels.

As the torsion pendulum will work in the final stage at a high vacuum the device made is enterely commanded from the exterior by means of magnetic fields generated by the polar pieces *33* and *34*. These are permanent magnets which create between the poles a magnetic field with $B=0.3$ T. These fields rotate the soft iron pieces *12* and *7* from Fig. 1. We found this value of B experimentally by setting the whole device from Fig. 1 between

the poles of a Weiss electromagnet. Thus we have determined the minimal value B_0 of B, for which we can rotate the pieces *7* and *12*. Then we have made the permanent magnets *33* and *34* which are to generate a field between the poles with $B > B_0$.

4. Conclusions. 1. The devices we have worked out allows the rotation in a horizontal plane and independently the vertical translation of an arbitrary load. These operations can be performed by means of the magnetic field, from the outside of the room containing the installation.

2. The devices can be thus used in other installations as well, except the one for which it was projected, mentioning that the parameters that characterise the rotation and the translation can be modified according to the necessities.

Received December 4, 1984

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REFERENCES

1. Roll P.G., Krotkov R., Dicke R.H., Annals of Physics (U.S.A.), 26, 442 (1964).
2. Брагинский В. Б., ЖЭТФ, 61, 873 (1971).
3. *** Contract de cercetare nr. *24 218*, ICPE–CNST şi Inst. polit. Iaşi.
4. Holland-Merten F.L., *Tehnica vidului*. Ed. tehh., Buc., 1958.
5. Antoniu I.S., *Bazele electrotehnicii*. Ed. did. şi ped., Buc., 1974.
6. Mangeron D., Irimiciuc N., *Curs de mecanică cu aplicaţii în inginerie*. Iaşi, 1974.

DISPOZITIV PENTRU REGLAREA ÎN VID A POZIŢIEI PENDULULUI DE TORSIUNE PENTRU VERIFICAREA PRINCIPIULUI ECHIVALENŢEI

(Rezumat)

Dispozitivul de reglare în vid a poziţiei pendulului de torsiune este o anexă la balanţa gravitaţională realizată în Institutul politehnic Iaşi, care are ca scop verificarea principiului echivalenţei. Deoarece instalaţia menţionată va lucra, în final, la un vid înaintat, dispozitivul realizat este comandat în întregime din exterior prin intermediul cîmpurilor magnetice.

Reglarea poziţiei pe verticală şi în plan orizontal se realizează independent.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

C.Z. 538.6 ; 621.778

EVALUAREA DIRECŢIEI OPTIME A FORŢEI DE LEVITAŢIE ŞI PROIECTAREA SEPARATOARELOR CU FEROFLUID

DE

GH. COMAN, GH. CĂLUGĂRU, DAVID ZOLER, C. COTAE şi I. STARPARU

1. Introducere. Interesul deosebit prezentat de necesitatea separării magnetice după densitate, avînd ca scop separarea materialelor neferomagnetice necesare în procesele tehnologice (din secţiile de turnătorie) de obţinere a aliajelor cu materiale de puritate ridicată, a condus la dezvoltarea metodelor de separare magnetice neconvenţionale.

Rezultate deosebite în această direcţie au fost obţinute folosind proprietatea de levitaţie a corpurilor neferomagnetice cufundate în ferofluide [1] ... [3], levitaţie care apare datorită forţei cu care ferofluidul, plasat într-un gradient de cîmp magnetic, acţionează asupra corpurilor neferomagnetice, în sens opus gradientului de cîmp magnetic.

În funcţie de orientarea forţei de levitaţie s-au propus diverse metode şi deci instalaţii de separare, astfel încît în unele metode, forţa de levitaţie este orientată pe direcţia verticală, în sens contrar forţei de gravitaţie [4], [5], în alte metode [6], [7] forţa de levitaţie acţionează în plan orizontal, separatorul funcţionînd ca un spectrograf de densitate sau [8], [9] cînd forţa de levitaţie este orientată pe o direcţie ce face un unghi γ cu orizontala (fig. 1). În ultimul caz (fig. 1) materialele ce urmează a fi separate se deplasează sub acţiunea forţei rezultante sub un unghi θ în raport cu orizontala, unghi care depinde de densitatea materialelor ce urmează a fi separate.

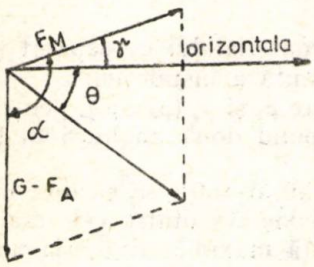


Fig. 1

În prezenta lucrare se prezintă condiţiile teoretice şi se stabileşte o relaţie teoretică de calcul pentru o separare optimă a materialelor neferomagnetice. Este prezentat modul de utilizare a consideraţiilor teoretice şi proiectul unui separator de laborator.

Sînt daţi parametrii numerici pentru separarea pereche a celor mai uzuale materiale neferomagnetice şi se determină constanta aparatului *a*, pentru modelul de laborator.

2. Aspecte teoretice privind evaluarea direcţiei optime a forţei de levitaţie. Este cunoscut că levitaţia de speţa întîi în ferofluide este dată de forţele care apar atunci cînd un corp de natură nemagnetică este scufundat într-un ferofluid aflat sub acţiunea unui cîmp magnetic neuniform [1].

Gradientul cîmpului magnetic acţionează pe o direcţie ce face un unghi γ cu orizontala (opus forţei magnetice de levitaţie, fig. 1) ; tangenta unghiului θ are expresia [10], [11]

$$\text{(1)} \qquad \operatorname{tg}\theta = \frac{F_G - F_M \sin\gamma}{F_M \cos\gamma},$$

unde F_G reprezintă forţa gravitaţională, diferenţa dintre greutatea corpului nemagnetic scufundat şi forţa arhimedică, considerînd că forţa de rezistenţă v scoasă este neglijabilă (întrucît V, η, şi r sînt foarte mici, $F_G = G - F_A$) ; F_M este forţa de levitaţie [1] ... [3], [9] ... [11] dată de relaţia

$$\text{(2)} \qquad \vec{F}_M = \mu_0 \vec{M}_F (\operatorname{grad} H).$$

Cînd cîmpul în care se găseşte ferofluidul este suficient a aduce ferofluidul la starea de saturaţie magnetică, în relaţia (2) M_F este înlocuită cu magnetizaţia de saturaţie a ferofluidului (M_s).

Ţinînd cont de (2), relaţia (1) capătă forma

$$\text{(3)} \qquad \operatorname{tg}\theta = \frac{g}{\mu_0 M_f (\operatorname{grad} H)} \frac{\rho - \rho_f}{\cos\gamma} - \operatorname{tg}\gamma,$$

unde ρ este densitatea materialului din care provin particulele solide ce urmează a fi separate, ρ_f — densitatea ferofluidului, iar M_f reprezintă magnetizaţia ferofluidului la cîmpul în care se găseşte. Dacă notăm

$$\text{(4)} \qquad a = \frac{g}{\mu_0 M_f (\operatorname{grad} H)},$$

relaţia (3) capătă forma

$$\text{(5)} \qquad \operatorname{tg}\theta = \frac{a(\rho - \rho_f)}{\cos\gamma} - \operatorname{tg}\gamma.$$

Se remarcă că a nu depinde de volumul materialului de separat şi nici de densitatea acestuia, reprezentînd o constantă a instalaţiei.

În cazul a două materiale cu densităţi diferite ρ_1 şi ρ_2 ($\rho_2 > \rho_1$), pentru un cîmp H constant, conform relaţiei (5) corespund două unghiuri θ_1 şi θ_2, cu diferenţa dintre ele $\Delta\theta = \theta_2 - \theta_1$.

Din condiţia de maxim a diferenţei $\Delta\theta$ [$d(\Delta\theta)/d\gamma = 0$], se găseşte un unghi γ optim (γ_0), pentru care unghiul de deflexie $\Delta\theta$ dintre cele două materiale este maxim iar separarea are o eficienţă maximă. Expresia unghiului optim de înclinare γ_0 obţinută este

$$\text{(6)} \qquad \sin\gamma_0 = \frac{a(\rho_1 + \rho_2 - 2\rho_f)}{1 + a^2(\rho_1 - \rho_f)(\rho_2 - \rho_f)},$$

cu condiţiile

$$\rho_1 < \rho_f + \frac{1}{a} \quad \text{şi} \quad \rho_2 < \rho_f + \frac{1}{a},$$

sau

$$\rho_1 > \rho_f + \frac{1}{a} \quad \text{şi} \quad \rho_2 > \rho_f + \frac{1}{a},$$

condiţii pentru care $\Delta\theta$=maxim, $\gamma = \gamma_0$ şi deci separarea are eficienţă maximă.

3. Parametrii optimi ai separatorului cu ferofluid. Ferofluidul, partea activă a separatorului, este dispus într-o incintă plasată între piesele polare, avînd forma întrefierului dintre acestea.

Ferofluidul utilizat în cadrul proiectării are o magnetizaţie de saturaţie $M_s = 2.10^4$ A/m.

Dependenţa $M_f = f(H)$ a ferofluidului este prezentată în fig. 2, avînd un factor demagnetizant la determinare $L = 2.10^{-2}$ [12].

În fig. 3 este prezentată schiţa separatorului proiectat, iar în fig. 4 configuraţia pieselor polare. Planul $OABC$ din sistemul de coordonate XY reprezintă o secţiune în incinta cu ferofluid, perpendiculară pe axa polilor (fig. 4).

Pentru o separare optimă, într-un separator este necesar a fi cunoscută zona de alimentare cu materialul de separat şi zonele (punctele) de evacuare a materialului separat la ieşirea din incintă. Necunoaşterea acestor condiţii conduce la o separare parţială a componentelor sau chiar la nesepararea lor.

Pentru un separator cu piese polare ca cele reprezentate în fig. 4 dar cu o orientare a planului XY ca cel din fig. 5, unde $OA = l$ şi $OC = L$, este necesar a calcula pentru fiecare γ_0 valorile lui θ_1 şi $\Delta\theta$ cu ajutorul relaţiei (5).

Din fig. 5 se constată că există un interval PQ pe latura OC în care punctul T (de alimentare) se poate deplasa, astfel încît punctele de evacuare (a materialelor separate) aflate pe latura AB sînt MN şi se pot deplasa pe intervalul AB. O separare eficientă se obţine atunci cînd punctele MN sînt depărtate, deci cînd l este mare.

Segmentele QC şi OP sînt determinate cu relaţiile

$$QC = l \operatorname{ctg}(\gamma_0 + \theta_1), \qquad OP = -l \operatorname{ctg}(\gamma_0 + \theta_1 + \Delta\theta). \tag{7}$$

Unghiul θ_1 se poate calcula cu ajutorul relaţiei (5), cînd se înlocuieşte unghiul γ cu γ_0 (unghiul optim pentru materialele de separat). În tabelul 1 sînt prezentaţi parametrii γ_0, θ_1, $\Delta\theta$, OP, QC, pentru diverse perechi de materiale, cînd valoarea coeficientului a este $a = 5.10^{-4}$ m³/kg, iar densitatea ferofluidului $\rho = 1{,}1.10^3$ kg/m³.

Instalaţia prezentată anterior a fost etalonată cu ajutorul unui teslametru cu sondă Hall de tip B.H. 32 în domeniul curenţilor de excitaţie de la 1 la 5 A. Etalonarea a fost efectuată pentru direcţia de lucru, direcţie perpendiculară pe liniile de cîmp magnetic, paralel cu direcţia gradientului de cîmp.

S-a măsurat dependenţa cîmpului magnetic de intensitatea curenţilor de excitaţie a electromagnetului, pentru două perechi de piese polare ($\alpha = 30°$, $\alpha = 45°$) cu unghiuri de deschidere diferite. Cîmpul magnetic a fost măsurat în zonele de cîmp maxim şi minim la limita pieselor polare cît şi la centrul acestora. A fost apoi evaluat parametrul a corespunzător tipului de piese polare şi variaţia acestuia cu valoarea curentului de excitaţie (fig. 6) pentru două ferofluide cu magnetizaţia de saturaţie diferită.

Se constată că la instalaţia proiectată se poate obţine valoarea considerată în calcul de 5.10^{-4} (m^3/kg) pentru parametrul instalaţiei a, pentru care a fost stabilit unghiul optim γ_0 şi unghiul de deflexie $\Delta\theta$ (tabelul 1).

Tabelul 1

	Zn	Cd	Cu	Mo	Ag	Pb	Hg
Cd	$\gamma=33,3$ $\Delta\theta=4,1$ $\theta_1=71,27$ $OP=0,34\,l$ $QC=0,26\,l$						
Cu	$\gamma=32,7$ $\Delta\theta=4,8$ $\theta_1=71,22$ $OP=0,33\,l$ $QC=0,25\,l$	29,2 0,6 75,1 0,26 l 0,25 l					
Mo	$\gamma=30,9$ $\Delta\theta=6,7$ $\theta_1=71,06$ $OP=0,33\,l$ $QC=0,21\,l$	27,3 2,8 74,97 0,26 l 0,21 l	26,8 2,2 75,61 0,26 l 0,21 l				
Ag	$\gamma=30,6$ $\Delta\theta=7,3$ $\theta_1=71,04$ $OP=0,34\,l$ $QC=0,2\,l$	26,5 3,2 74,95 0,26 l 0,2 l	26,4 2,6 75,58 0,26 l 0,21 l	24,4 0,4 77,58 0,22 l 0,21 l			
Pb	$\gamma=27,8$ $\Delta\theta=10,1$ $\theta_1=70,85$ $OP=0,34\,l$ $QC=0,15\,l$	24,1 4,2 74,8 0,23 l 0,16 l	23,5 3,6 75,44 0,22 l 0,16 l	23,4 1,5 77,53 0,22 l 0,19 l	23,0 1,0 77,94 0,21 l 0,19 l		
Hg	$\gamma=27,8$ $\Delta\theta=10,1$ $\theta_1=70,85$ $OP=0,34\,l$ $QC=0,15\,l$	24,1 6,3 74,8 0,27 l 0,16 l	23,5 5,7 75,44 0,26 l 0,16 l	21,5 3,5 77,46 0,22 l 0,16 l	21,1 3,1 77,86 0,21 l 0,16 l	18,3 0,2 78,8 0,13 l 0,12 l	
W	$\gamma=25,4$ $\Delta\theta=13,3$ $\theta_1=70,74$ $OP=0,35\,l$ $QC=0,1\,l$	21,4 9,2 74,7 0,27 l 0,11 l	20,8 8,6 75,34 0,26 l 0,1 l	18,6 6,5 77,37 0,22 l 0,1 l	18,4 6,0 77,7 0,21 l 0,1 l	15,5 3,1 78,76 0,13 l 0,07 l	15,4 2,9 80,8 0,16 l 0,1 l

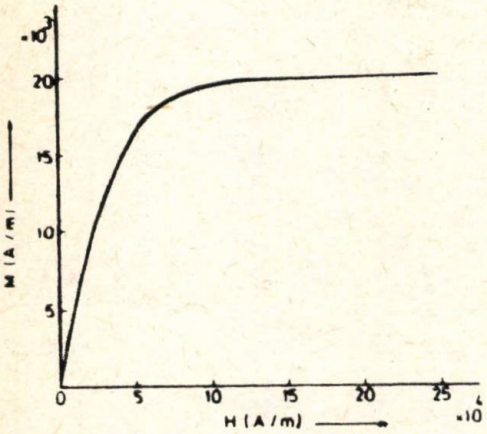


Fig. 2

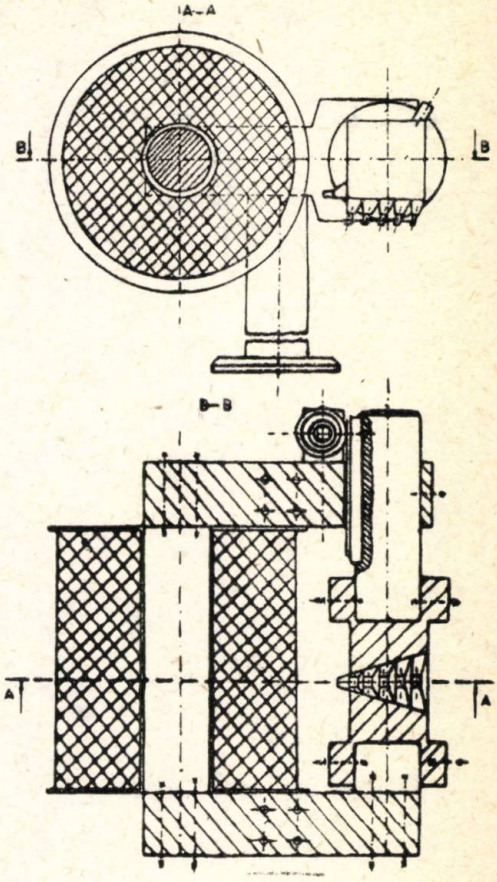


Fig. 3

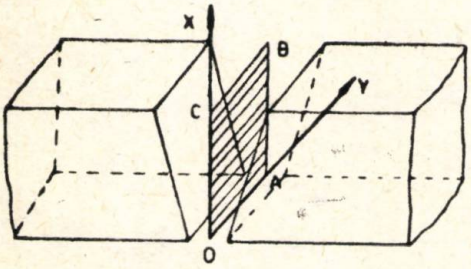


Fig. 4

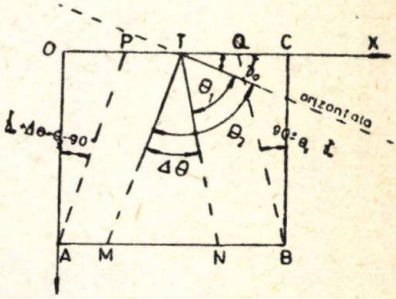
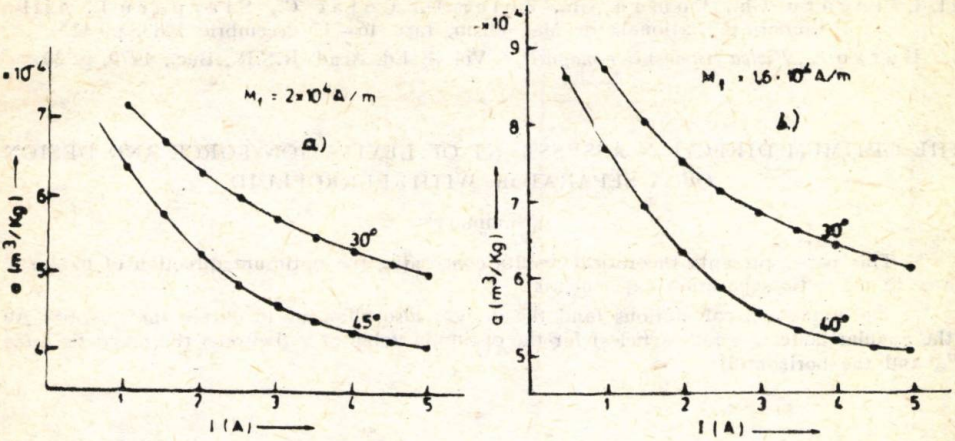


Fig. 5



a Fig. 6 *b*

4. Concluzii. 1. Pentru fiecare pereche de materiale există un unghi optim γ_0, pentru care unghiul de deflexie $\Delta\theta$ are valoarea maximă.

2. Pentru fiecare pereche de materiale există o zonă optimă de turnare care depinde de parametrii constructivi (l şi L) ai separatorului şi de valoarea parametrului a.

3. O separare totală, a unui amestec de materiale de densităţi diferite, se poate face succesiv pe grupe restrînse de densităţi.

4. Instalaţia proiectată realizează un coeficient a egal cu cel propus în calcule şi poate fi utilizată pentru separarea materialelor prezentate în tabelul 1.

Primită la 20 decembrie 1984

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BIBLIOGRAFIE

1. Luca E., Călugăru Gh., Bădescu R., Cotae C., Bădescu V., *Ferofluidele şi aplicaţiile lor industriale*. Ed. tehn., Buc., 1978, p. 336.
2. Călugăru Gh., Cotae C., *Lichide magnetice*. Ed. ştiin. şi enciclop., Buc., 1978, p. 98.
3. Rezlescu N., Bădescu V., Bradu E.B., Iacob Gh., *Principiile separării magnetice a materialelor*. Ed. Acad. R.S.R., Buc., 1984, p. 220.
4. Kaiser R., U.S. Patent, *3 483 968*, Dec. 1969.
5. Rosensweig R.E., U.S. Patent, *3 483 969*, Dec. 1969.
6. Юшан В., Хомородян Л., Магнитная гидродинамика, *3*, 133 (1982).
7. Iuşan V., Homorodean L., Moloescu I., Muntean V., A III-a Conferinţă Naţională de Magnetism, Iaşi, 16—17 decembrie 1983, p. 431.
8. Khalafalla S.E., Reimers G.W., Separation Science, **8** (12), 161 (1970).
9. Khalafalla S.E., IEEE Trans. on Magnetics, **MAG-12**, *5*, 455 (1976).
10. Zoler D., Segal A., Călugăru Gh., Compt. Rend. de l'Acad. Bulgare des Sci., **36**, *5*, 591 (1983).

11. Călugăru Gh., Coman Gh., Zoler D., Cotae C., Starparu I., A III-a Conferinţă Naţională de Magnetism, Iaşi, 1[illegible]–17 decembrie 1983, p. 425.
12. Burzo E., *Fizica fenomenelor magnetice*. Vol. 1, Ed. Acad. R.S.R., Buc., 1979, p. 30.

THE OPTIMUM DIRECTION ASSESSMENT OF LEVITATION FORCE AND DESIGN OF A SEPARATOR WITH FERROFLUID

(Summary)

This paper presents theoretical results concerning the optimum direction of magnetic force in magnetic separation experiments.

The numerical calculations (and the design) also allow us to obtain the quantity $\Delta\theta$ (the angular deflection for particles) for the optimum value of γ (between the magnetic force F_m and the horizontal).

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul XXX (XXXIV), Fasc. 1—4, 1984
Secţia I
MECANICĂ

D.C. 535.215

ON THE PHOTOELECTRIC PROPERTIES OF Al-Chl-Cu SYSTEM

BY

GH. ZET, MAGDA GHERGHEL, EMILIANA GHERASIMESCU,
I. ROŞCA and IOANA MIHAI

1. Some Considerations Concerning the Photosynthetic Pigments. The studies dealing with the photosynthetic pigments [1] ... [9], proved that these organic compounds possess two important properties necessary for a material used in the preparation of an efficient photovoltaic device, namely a strong absorption in the visible range (where the solar radiant energy is maximum) and an activation semiconduction energy of 1.2 to 1.65 eV, i.e. within the range 1 ... 2 eV, where the calculated efficiencies of the photovoltaic devices are maximum [1] .. [3], [5], [8], [9].

The solar energy is efficiently used by the nature ; by means of the photosynthesis it penetrates the photosynthetic apparatus of the superior plants and algas by means of the photosynthetic pigments, known as chlorophyls (complex chromolipoproteids made of bipolar molecules) [4], [6] ... [8]. During the photosynthesis, the chlorophyl plays the role of photoexcitation collectors — „antenna chlorophyl" (chlorophyl a_{683}, a_{672}), that of electron donor in primary photosynthetic reactions — „reaction center chlorophyl" (chlorophyl a_{700}, a_{683}) and that of a protecting screen against the utraviolet rays (chlorophyl a_{672}, chlorophyl b and carotene pigments) [4], [5], [7], [8].

The structural organization of the chlorophyl molecule in the photosynthetic membrane is made so that it allows perhaps an efficient transfer of the photoexcitation from the „antenna molecules" to the „reaction center molecules" [5], [8]. On the other hand, the presence of the double conjugated bounds in the chlorophyl molecule implies the existence of the mobile π electrons that can easily make transitions from one molecule to another, this explaining the electric conduction mechanism in double bounded organic compounds, where the activation energy for the electronic transfer depends on the number of π electrons of the corresponding molecule [7], [9].

During the last years, the study of the factors qualitatively determining the solar energy conversion efficiency in the photosynthetic process, the elucidation of the excitation transfer in the photosynthetic cell and the study of the photovoltaic properties of different kinds of chlorophyls presents a special interest for the future energetics, both theoretically and practically. Thus, relatively recent researches [1], [3] deal with the

preparation of microcrystalline chlorophyl and the study of its photoconductive properties. Tang and Albrecht obtained thin microcrystalline chlorophyl *a* layers presenting photoeffects, a high conversion efficiency and remarkable electric and optic properties [1] ... [3].

Taking into account the high absorption in the visible range in the photosynthetic pigments and the photoconducting properties found in microcrystalline chlorophyl layers [1] ... [5], we proposed the realization of a photovoltaic system $Metal_1$-Chlorophyl-$Metal_2$ of Al-Chl-Cu type and the study of its photoelectric behaviour.

2. Experimental Procedure. Results. With the view of preparing the Al-Chl-Cu photovoltaic system, we first made thin layers of chlorophyl *a* and *a*+*b* respectively, by three methods, namely: *A*—spray method; *B*—„dipping method" that takes into account the different solubility of the photosynthetic pigments in different organic solvents and their property to volatilize at very low temperature (30 ... 60°C); *C*—electrodeposition method.

A. The chlorophyl *a* was previously extracted from fresh leaves (of lucerne and stinging nettle) by the Machlis and Torrey method [6], [7] using acetone mixed with distiled water, petroleum ether and ethyl ether as solvents [5] ... [7]. After removing the hydroacetate layer, the extract of chlorophyl *a* in ethyl ether was separated, with a concentration of 16 mg/ml and pulverized at a pressure of 1.4 10^5 Pa, on glass substrates (well polished and treated with aqua regia) of 3 ×7 cm and 2.5 ×2.5 cm respectively, on which thin Al layer was deposited by thermal vacuum evaporation, the Al layer having a central transparent region of 1.5 ×1.5 cm.

Aiming to obtain the microcrystalline structure at the thin chlorophyl *a* layers that we prepared by spray method, the samples were hold for 1 ... 2 minutes in water vapour atmosphere, since „in vivo" the chlorophyl represents (after Fong and co.) a complex polymer interacting with the water $(Chl_a-2H_2O)_n$ [5], [8]. At the same time, a series of studies [5], [8], [11] emphasized the increased photoconductivity of the chlorophyl in the presence of water vapours, the action spectrum of chlorophyl photoconductivity shifting toward the red end when the pigment is activated with water [4], [5], [8].

Over the thin sprayed chlorophyl layer, a thin Cu layer was vacuum deposited, with a thickness between 0.5 and 5.0 μm for a great number of samples, thus being obtained the Al-Chl_a-Cu structure (Fig. 1). The average thickness of the sprayed chlorophyl *a* layer was 20 μm for all the samples and their average resistivity at room temperature was (1.5 ...2)10^{14} Ωcm.

B. The thin chlorophyl layers made by dipping method were obtained by dipping the glass samples with the Al electrodes deposited of them (by thermal vacuum evaporation at 10^{-5} Torrs) in the chlorophyl *a* extract with a mean concentration of 10 mg/ml, so that by using masks adequate to the substrates, the sample surface was bathed by a thin uniform layer of chlorophyl extract. By evaporating the solvent (acetone, petroleum ether, ethyl ether) a chlorophyl *a* layer is formed on the surface of the Al electrode (having a central transparent area) more uniform than that obtained by the spray method.

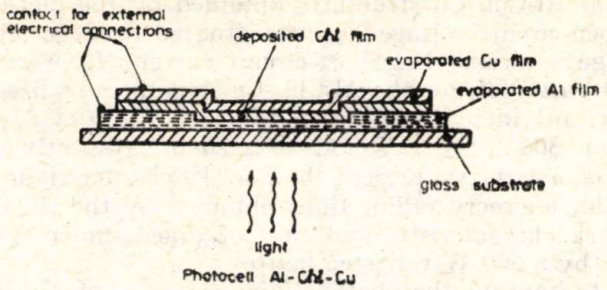


Fig. 1

The thickness of the chlorophyl *a* layers obtained by this methods ranged between 15 and 25 μm depending on the concentration of the extract we used ; their average resistivity was situated between $1.5\ 10^{11} \ldots 3.5\ 10^{14}$ Ω cm. In this case too the samples were kept for 1 ... 2 minutes in water vapour atmosphere.

As in the case *A*, above the microcrystalline chlorophyl *a* films, thin Cu films were thermally evaporated, with an average thickness of 0.5 ... 5μm for all the samples. And thus we obtained the Al-Chl_a-Cu structure.

C. In order to obtain thin chlorophyl layer by electrodeposition [1] ... [3], [5], [11], the chlorophyl microcrystals were previously obtained by evaporating a concentrated solution (20 mg/ml) of chlorophyl *a* extract in ethyl ether on the water surface. The gigment microcrystals were suspended in a solution of isooctane, representing the intermedium during the electrodeposition process. The cathode consisted of glass samples with Al electrodes (with the same planar geometry as in the two previous procedures, and the anode was made of well polished Al sheet. The two electrodes were immersed in the complex : chlorophyl suspension/isooctane solution, being separated by 5 ... 15 mm thick raylon spacers.

A strong electric field of 1 500 ... 2 500 V/cm was applied between the electrodes. Thin uniform microcrystalline films were obtained with mean thicknesses of 10 ... 20 μm and resistivity of $5\ 10^{7} \ldots 3\ 10^{11}$ Ωcm, depending on the deposition time (3 ... 8 minutes) and the solution concentration, situated between 10 and 25 mg chlorophyl/50 cm^3 isooctane. Finally, Cu electrodes were vacuum deposited, the film thickness being of about 2.5 μm and thus was prepared the Al-Chl-Cu system as a whole (Fig. 1).

The layer resistivity was determined with the EG-402 probe multimeter [9] and the thickness was measured with Linnik interference microscope.

The electric measurements for Al-Chl-Cu devices prepared by us by means of the three previously described methods were made using a series circuit presented in Fig. 2.

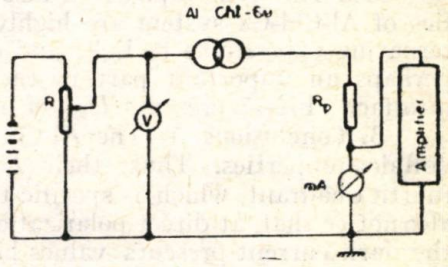


Fig. 2

For all the Al-Chl$_a$-Cu structures obtained by the methods *A* and *B* the average open-circuit voltage U_{0c} was situated between 50 and 300 mV and the average values of the short-circuit current I_{sc} were ranging between 0.5 and 10 nA. Yet, for the Al-Chl$_a$-Cu structures realized by the method *C* a significant increase was seen for both U_{0c} and I_{sc}, their values ranging between 300 ... 500 mV and 10 ... 20 nA respectively.

In Figs. 3*a*, *b* and *c* we present the $I=f(V)$ characteristic of Al-Chl$_a$-Cu device (with Chl$_a$ microcrystalline films obtained by the three procedures), namely the dark characteristic and that obtained under continuous illumination given by a 500 W tungsten lamp.

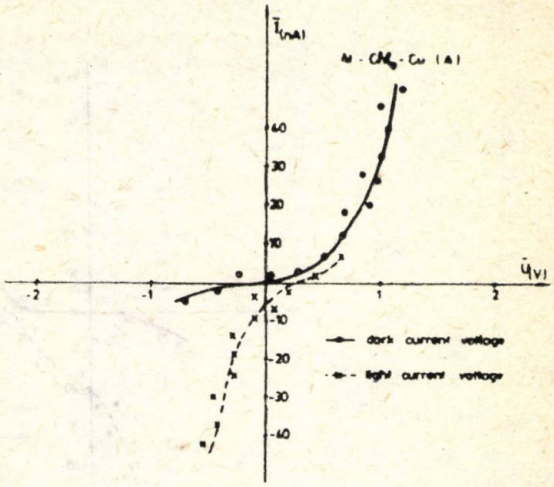
In order to improve the photovoltaic properties of Al-Chl$_a$-Cu system, we have also studied the possibility of obtaining a combined pigment thin film of the Chl$_{(a+b+\text{carotene})}$ type taking into account both the fact that „in vivo" the chlorophyl *a* pigment coexists with the chlorophyl *b* pigment, the latter playing a part in the efficient collection and transfer of the light energy to the photosynthesis reaction center [7], [8], and that in the plant the carotene prevents the photooxidization of chlorophyl [4], [5]. Thus, we elaborate a faster and more simple method for extracting and separating the chlorophyl *a* and *b* pigments and the carotene from the vegetal matters, by using the normal hexane, cyclohexane or isooctane for the complex Chl suspension/solvent, the mean concentration of the chlorophyl solution being 20 mg/50 ml solvent (80% chlorophyl *a*, 10% chlorophyl *b* and 10% carotene).

The Al-Chl$_{a+b+\text{carotene}}$ -Cu device prepared under the same working conditions as Al-Chl$_a$-Cu, presented a remarkable improvement of the photoeffect, namely an amplification of the photocurrents and photovoltages, the average $\bar{U}_{0c}$ (in the case of all the Chl$_{a+b}$ films obtained by the methods *A* and *B* and *C*) being 600 mV and the average $\bar{I}_{sc}$, 20 nA. Fig. 4 shows the $I=f(V)$ characteristic for Al-Chl$_{a+b+\text{carotene}}$-Cu systems.

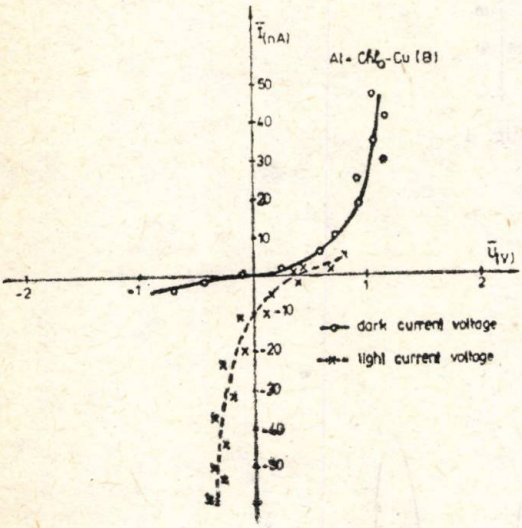
The study of the spectral characteristic $U_{0c}=f(\lambda)$ for the two types of devices made evident that the action spectrum of the photoconduction closely follows the absorption spectrum of microcristalline chlorophyls [1] ... [3], [5], [7], [8], a shift of the red maximum toward the higher wavelength ($\lambda=745$ nm, $\lambda=780$ nm) being specific. In Figs. 5 and 6 we present the spectral characteristics $U_{0c}=f(\lambda)$ for Al-Chl$_a$-Cu and Al-Chl$_{a+b+\text{carotene}}$-Cu devices respectively, the last one presenting a significant improvement of the optic and electric properties, hence a widening of the photoconductive region (see Figs. 5 and 6).

Our research emphasized also the fact that the photovoltaic properties of Al-Chl-Cu system are highly influenced by the duration of the alternating expositions to light and darkness, the illumination time playing perhaps an important part in the processes produced at Al-Chl-/Chl-Cu interface. Fig. 7 presents U_{0c} vs. illumination time, $U_{0c}=f(t_{\text{illum}})$.

3. Conclusions. 1. The Al-Chl-Cu devices present remarkable photovoltaic properties. Thus, their $I=f(V)$ characteristics pass through the fourth quadrant, which is specific to all the photovoltaic devices. One can also notice that, at direct polarization of this device (Al electrode as cathode) the dark current presents values higher than at reversed polarization, the device possessing rectifying properties. We consider that this photoelectric


Ī(nA)
Al - CAl₂ - Cu (A)
Ū(V)
dark current voltage
light current voltage

a


Ī(nA)
Al - CAl₂ - Cu (B)
Ū(V)
dark current voltage
light current voltage

b

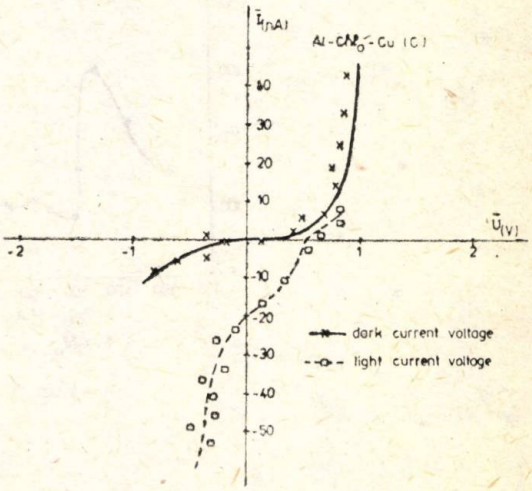

Ī(nA)
Al - CAl₂ - Cu (C)
Ū(V)
dark current voltage
light current voltage

Fig. 3 *c*

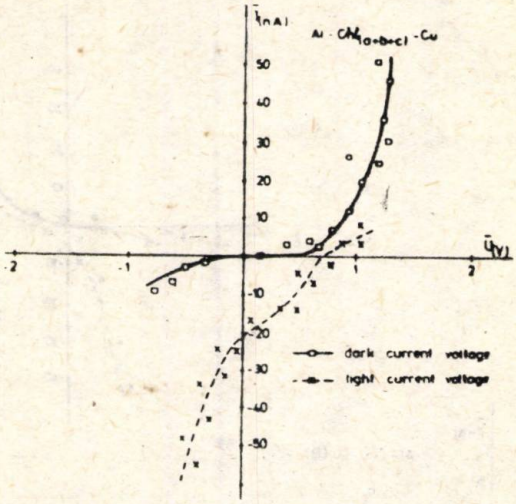

Ī(nA)
Al - ChlO+D+C1 - Cu
50
40
30
20
10
-10
-20
-30
-40
-50
-2
-1
1
2
Ū(V)
dark current voltage
light current voltage

Fig. 4

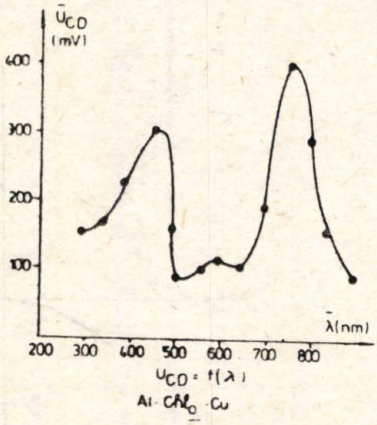

Ū_CD
(mV)
400
300
200
100
λ(nm)
200
300
400
500
600
700
800
U_CD = f(λ)
Al - Chl_O - Cu

Fig. 5

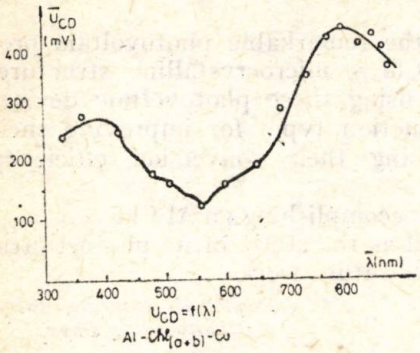


Fig. 6

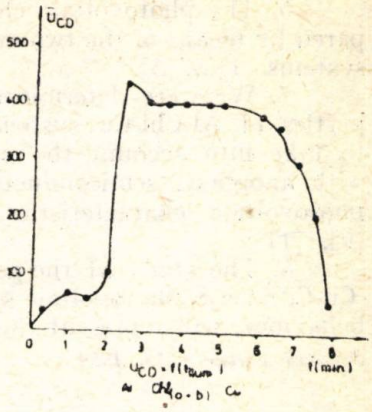


Fig. 7

behaviour of Al-Chl-Cu systems could be justified taking into account the *p*-type semi-conducting properties of the chlorophyl [4], [5], [8], and the active photo-conducting region at the contact between the pigment film and Al electrode.

2. Both the values of Chl layer resistivity and the photovoltaic characteristics proved that Al-Chl-Cu systems with pigment layer obtained by the electrodeposition (method *C*) had obviously improved photovoltaic properties, as compared to *A* or *B* devices. We explain this by the nonuniform thickness of chlorophyl films obtained by the methods *A* and *B* and their poor adherence with the substrate. In exchange, we found at these layers an increased photoconductivity when treating them with water vapours for 3 ... 5 minutes.

3. The photovoltaic properties of Al-$Chl_{a+b+carotene}$-Cu systems show their better photoelectric qualities, as compared to those of Al-Chl_a-Cu systems. We suppose this is due to the capacity of chlorophyl *b* to intercept photons and transfer them to the chlorophyl a_{683} by molecular induction or by inductive resonance [6], [8], [9], the excitation energetic levels of π electrons from the photosynthetic pigment molecules depending on the wavelength of the phonon that entered in excitation resonance with the pigment molecule [2], [3], [6] ... [8]. At the same time, the carotene layer interposed between Chl_{a+b} and Cu layers prevents the Al-Chl-Cu system from photooxidation.

4. The specific influence of the illumination time of the samples al Al-Chl junction was noticed, the optimum illumination time being of 3 ... 5 minutes. An illumination time exceeding 5 minutes has negative influence on the system photoelectric properties.

5. It is also worth noticing that for the both types of systems, a Cu layer thicker than 5 μm results in an appreciable weakening of their photovoltaic properties, the optimum thickness of Cu film ranging between 1.5 and 2 μm.

6. The photovoltaic characteristic of Al-Chl-Cu devices that we prepared by means of the two methods are comparable with those of Al-Chl-Hg systems [1] ... [5].

7. We were determined by both, the remarkable photovoltaic properties of Al-Chl-Cu systems and the Chl_{a+b} microcrystalline structure, to take into account the possibility of using these photovoltaic devices with anorganic semiconductors (multi-junction type) for improving their photovoltaic characteristics and increasing their conversion efficiency [9], [11].

8. The study of the possibilities of accomplishing an Al-$Chl_{a+b+carotene}$-Cu-CdS-Cu_xS photovoltaic system, as well as the study of its photoelectric behaviour, will represent the object of our future work.

Received December 13, 1984

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REFERENCES

1. Tang C.W., Albrecht A.C., Mol. Liq. Cryst., **25**, 53 (1974).
2. Tang C.W., Albrecht A.C., J. Chem. Phys., **62**, 2 139 (1975).
3. Tang C.W., Albrecht A.C., J. Chem. Phys., **63**, 953 (1975).
4. Gheorghe V., Ţugulea L., Bejan A., Al IV-lea Simpozion Naţional de Biofizică, 1979.
5. Gheorghe V., Ţugulea L, *Proprietăţile optice şi electrice ale clorofilelor*. An. Univ. Bucureşti, s. Fizică, **XXVI**, 178 (1977).
6. Boldor O., Raianu O., *Fiziologia plantelor*. Ed. did. şi ped., Buc., 1982.
7. Milică I. şi colab., *Fiziologia vegetală*. Ed. did. şi ped., Buc., 1982.
8. Ştirban M., *Procese primare în fotosinteză*. Ed. „Dacia", Cluj-Napoca, 1981.
9. Rusu I. Gh., *Semiconductori organici*. Ed. tehn., Buc., 1983.
10. Cardon F., Gomes W.P., Dekeyser W., *Photovoltaic and Photoelectrochemical Solar Energy Conversion*. Plenum Press, New York, 1981.
11. Gherghel M., Zet Gh., Gherasimescu E. şi colab., Al II-lea Colocviu Naţional de Fizica şi Tehnologia Materialelor Cristaline şi Amorfe, Iaşi, 1984, p. 81.

ASUPRA PROPRIETĂŢILOR FOTOELECTRICE ALE SISTEMULUI Al-Chl-Cu

(Rezumat)

Se prezintă atît rezultatele experimentale privind realizarea unui sistem fotovoltaic de tipul Al-Chl-Cu, cît şi concluziile privitoare la studiul comportării fotoeletrice a acestui sistem. Astfel, cercetările experimentale privind caracteristicile fotovoltaice la acest tip de dispozitiv evidenţiază faptul că sistemul Al-Chl-Cu prezintă proprietăţi fotoelectrice considerabile, ceea ce ar sugera utilizarea centrilor de reacţie fotosintetici în realizarea unor multijoncţiuni fotoconductoare, pe bază de substanţe anorganice, în scopul de a mări eficienţa acestora.

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